ECONOMIC PRODUCTION QUANTITY MODELS WITH VARIABLE PRODUCTION RATES

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SYSTEMS ENGINEERING

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This thesis, written by **SYED NAVEED KHURSHEED** under the direction of his Thesis Advisor and approved by his Thesis Committee, has been presented to and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN SYSTEMS ENGINEERING.**

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Dedicated

to

My Dear Parents

and Brothers

Acknowledgements

All praise is for ALLAH(SWT), the most compassionate, the most merciful. May peace and blessings be upon prophet Muhammed(PBUH), his family and companions. I thank almighty ALLAH(SWT) for giving me the knowledge and patience to complete this work.

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THESIS ABSTRACT

Name: SYED NAVEED KHURSHEED

Title: Economic Production Quantity Models with Variable Production Rates

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This thesis focuses on Economic Production Quantity problem. The EPQ problem is concerned with the selection of the optimal production time for a production process. The literature in the area of EPQ focused more on either the selection of production rate or the production time. We have developed a mathematical model for the selection of optimal production time in case the production rate shifts to a value lower than the starting value for the case when this time to shift is deterministic. The second model is developed for the case when this time to shift is a random variable with known general distribution. The third model is an extension of the second model to include the possibility of restoration of the system to the original production rate in case a shift is detected during process inspection. The fourth model includes the possibility of preventive maintenance to reduce the effective age of the system to decrease the probability of shift. The fifth and final model extends the general model by relaxing the assumption of deterministic production rate after the shift. Results show that for the case where the production rate shifts, the expected cost per unit time will be forced higher and also the system is forced to operate for longer periods of time. The thesis is concluded by suggesting a number of recommendations for future research.

Keywords: *Economic Modeling, Engineering Management, Production and Inventory Systems, Variable production rate, Preventive Maintenance.*

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تركّز هذه الرّسالة على مسألة كمّيّة الإنتاج الاقتصاديّة (EPQ). هذه المسألة معنيّة باختيار وقت الإنتاج الأفضل لعمليات الإنتاج . الأدب المنشور في مجال(EPQ) يركّز أكثر إمّا على اختيار معدّل الإنتاج أو وقت الإنتاج .في هذه الرّسالة قمنا بتطوير نموذج رياضيّ لاختيار وقت الإنتاج الأفضل للحالة التي فيها يتغير معدّل الإنتاج إلى قيمة أقل من القيمة الابتدائية عندما يكون هذا الوقت محددا . النّموذج الثّاني ُطَوَّر للحالة التي يكون فيها هذا الوقت متغيّرا عشوائياً بتوزيع عامّ معروف . النّموذج الثّالث امتداد للنّموذج الثّاني لتضمين احتمال اعادة النّظام إلى معدّل الإنتاج الأصليّ إذا تم اكتشاف التغير في أثناء فحص العمليّة . يتضمّن النّموذج الرّابع إمكانيّة الصّيانة الوقائيّة لتقليل العمر المؤثّر للنظام لتخفيض احتمال التغير. النّموذج الخامس و الأخير يمددّ النّموذج العامّ بإراحة فرض معدّل الإنتاج المحددّ بعد التغير . أظهرت النّتائج أن التّكلفة المتوقعة لكلّ وحدة سترتغعً للحالة التي يتغير فيها معدّل الإنتاج, و أيضًا سيُجْبَر النّظام على أن يشغل لفترات وقت أطول .ختمت الرّسالة باقتراح عدد من التّوصيات للبحث المستقبليّ.

درجةالماجستيرفي العلوم
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Nomenclature

In developing the models through out this thesis, the following abbreviations will be used.

P_1	=	Production rate at the beginning of the cycle,
P_2	=	Production rate after the shift,
D	=	Demand Rate,
I_1	=	Inventory level at the time of shift,
I_2	=	Inventory level at the end of the production cycle,
t	=	Known & deterministic time at which the process shifts from higher to lower production rate.
t_P	=	Total production time,
S	=	Setup cost,
h	=	Carrying cost per item per unit time,
a	=	The increase in unit machining cost due to increase in the production rate,
b	=	the per unit cost of running the machine independent of the production rate including labor and energy costs.
α	=	Cost of buying an item from outside,
C_m	=	Production cost per unit

C_p	=	Production cost per unit per unit time
C_l	=	Lost production cost per unit
π	=	Cost of bad quality items per unit
v	=	Cost of each inspection,
R_0	=	Fixed restoration cost associated with each restoration,
R_1	=	Variable restoration cost, dependent on detection delay,
C_{pm}	=	Cost of each PM activity,
γ	=	Ratio of the overtime unit production cost to the original unit production cost before the shift.
$f(t)$	=	The probability density function of the time to shift.
$g(P)$	=	Probability density function of P_2,
n	=	Number of inspections done in a cycle,
t_n	=	Total production time in case of restoration,
h_j	=	Length of the j^{th} interval,
t_j	=	Time elapsed from the beginning of production to the end of j^{th} interval,
y_j	=	The effective age of the system prior to the j^{th} PM,
b_j	=	The amount by which the age of the system is reduced by the j^{th} PM,
w_j	=	The effective age of the system after j^{th} PM.

I_j	=	Inventory level at the end of j^{th} interval,
α	=	Fraction of items produced of sub-standard quality,
λ	=	Rate of the shift.
TC_1	=	Total cost per cycle in case there is a shift,
TC_2	=	Total cost per cycle in case of no shift,
CL_1	=	Total cycle length in case there is a shift,
CL_2	=	Total cycle length in case of no shift,
TC	=	Total cost per cycle,
CL	=	Total cycle time,
$ETC(t_P)$	=	Expected total cost per unit time,

Chapter 1

Introduction

1.1 Overview

An implicit assumption to the simple EOQ model is that the items are obtained from an outside supplier. When this is the case, it is reasonable to assume that the entire lot is delivered at the same time or in simple words we are assuming that the production rate is infinite. However, if the units are produced internally then the assumption of infinite production rate has to be revised and the case of finite production rate has to be considered. This assumption of finite production rate gives rise to the **Economic Production Quantity** (EPQ) model.

Determining the Economical Production Quantity under various conditions has been of major interest to researchers. Recently, considerable effort has been devoted to extending the classical EPQ model to address various practical considerations.

Typical models for determining the EPQ assume perfect product quality and perfect production processes. However, product quality is usually dependent on the operating state of a production process that is subject to deterioration or failure due to the occurrence of some assignable cause. The process usually starts in an in-control state and produces items of acceptable quality. After some random point in time, the process may deteriorate and shift to an out-of-control state due to the occurrence of an assignable cause (for example due to aging, tool wear, overheating or vibration) and from that point on, the process produces non-conforming items.

Another underlying assumption of the EPQ model is that product quality is perfect and does not deteriorate with time. However, even if the supply system operates perfectly, defects may arise while the product is held in inventory. Perishable products, such as food, medicines, certain chemicals, radioactive materials, etc, because of their physical nature, systematically deteriorate over time. In such cases, on-hand inventory is depleted not only by demand but also by deterioration.

1.2 Effect of deteriorating process on the system

Process deterioration either increases operating costs or increases the probability of failure, where as product deterioration decreases the on-hand inventory level or decreases the customers' demand.

Deteriorating processes affect the system in the following manner:

- decrease the quality of the items produced,
- cause production stoppage,
- change the production rate.

The first two of the above are well studied, but the last one has not been adequately addressed. Our main purpose here is to explore the literature available on the variable production rate and to emphasize the importance of change of production rates with regard to the Economic Production Quantity model.

We intend to develop a basic model that can be used for the analysis of variable production rate problem keeping the time to shift as deterministic and later on extending it to accommodate the random shift following general distribution. Study the effects of integrating the inspection scheme to go along with it and later on the effects of performing preventive maintenance. We generalize the model to the case where the production rate after the shift is generally distributed, present numerical examples for all models to highlight the performance of the models.

Before further discussion a brief definitions of the terms used, is given.

1.3 Terms and Terminology

1. **Inventory:** Inventory is defined as the raw material, semi-finished parts and assemblies, and finished goods that are in the production system at any point

in time. Inventory serves as buffer between stages of production system and between the production system and customers. Inventory cost is the sum total of ordering cost or fixed cost (cost of replenishing inventory), holding cost (cost of carrying inventory), shortage cost (penalty on incurring shortage).

2. **Setup cost:** Every time we begin production, we incur some fixed cost which includes changing tools, oiling the machine etc., all these costs come under the head of machine setup for production run and all cost that do not vary with the size of production.

3. **Production cost:** The unit production cost will be taken to have a convex shape,(i.e., unit production cost is assumed to be convex in production rate) as different production rates result in different production costs. As the production rate is increased, some costs such as the labor and energy costs are spread over more units while per unit tool/die costs decrease. The net result is that unit production cost decreases until an ideal design production rate of the machine is reached, beyond which unit production cost increases. such as $aP + \frac{b}{P}$. The unit production cost is based on the following observations:

    - As the production rate is increased the labor and energy costs are spread over a large number of units.
    - There is a production rate at which the per unit tool cost is minimum. As the production rate is increased beyond that rate, the variable per

unit machining cost increases because of excessive tool wear.

4. **Holding cost:** The inventory carrying cost can be broken down into several components viz. (i) the *opportunity cost* of money being tied up in inventory, (ii) *storage and space charges*, representing the cost of providing storage space, as well as its cost of maintenance.

5. **Restoration:** Restoration is defined as bringing production system back to its natural state. If in inspection, the system is found in a state where it is producing low quality items than it is desired/desinged to (out-of-control state), repair work is done on the system and it is brought back to its natural state. Usually it is assumed this type of restoration gets system back to work without decreasing its chances to go back to out-of-control state.

6. **Inspection:** Inspection for the purpose of deciding whether the process is in-control or out-of-control is carried on at many points in the manufacturing cycle, we will consider equal integrated hazard rate inspections, other schemes include equally spaced and equal hazard rate etc.

1.4 Thesis Objectives

The objective of this thesis is to consider the variable production rate in the context of the Economic Production Quantity models. The specific objectives set are:

1. Basic model will be developed and analyzed for the variable production rate problem for the case where the time at which the shift from one to the other production rate occurs is deterministic and the possibility of out-sourcing the difference caused due to the shift in production rate will be considered.

2. The above model will be extended for the case when the time to shift follows general distribution.

3. Second objective will be further extended to consider the possibility of restorations alongside inspections to bring back the system to its original production rate.

4. The above model will be extended to include the possibility of performing preventive maintenance.

5. The assumption of the known production rate after the shift will be relaxed. It will be assumed that after some random time, the production rate change to a random value following general distribution.

1.5 Thesis Organization

The remainder of this thesis is organized as follows. Related research articles are reviewed and classified in the next chapter as well as the basic problem of variable production rates with reference to EPQ model will be defined and its scope identified. Chapter 3 deals with the deterministic time to shift between production rates. In chapter 4 we develop and analyze the model for random time to shift giving numerical examples and sensitivity analysis and chapter 5 presents the extension of 2^{nd} model to incorporate overtime production. In chapter 6, we present a model that incorporates the possibility of inspection and restoration to bring the system back to original condition (in case a shift is detected). In chapter 7, we extend the model developed in chapter 6 to incorporate preventive maintenance of the system. In chapter 8, we present a model where the production rate after the shift is also a random variable following general distribution between two certain values. Finally conclusions and directions for possible future research are presented in chapter 9.

Chapter 2

Literature Review and Problem Definition

2.1 Introduction

In this section we will review the literature available on the problem of variable production rate in EPQ models and its effect on the overall cost. Different researchers have studied the problem from different perspectives. Before we classify our review it would be suitable to shed some light on the basis of this classification. We have classified our models on the basis of the points of emphasis that the researcher have given in their studies. Broadly the work can be classified as:

- Variable Production Models

  The production rate has been treated as a decision variable.

- Cutting Speed Models

  The focus is on determining the optimal cutting speed (production rate change due to the change in cutting speeds)

- Shelf Life Models

  The problem is to choose the production rate under shelf life constraint (i.e. to account for product deterioration)

- Other Related Models

  The models that do not fit in any of the above categories will be discussed here.

2.2 Literature Review

The bulk of the literature available is related to the EPQ models. The classical economic production lot size model assumes a constant production rate that is predetermined, inflexible and produce items of perfect quality. Recent models have relaxed the inflexible production rate assumption. Production rates in many cases such as orders filled by machine can be changed. Moreover, unit production cost and process quality depend on the production rate. Most of the models attempt to make a trade-off between the productivity losses from making too small batches and the opportunity cost of tying up capital in inventory as large batches.

2.2.1 Variable Production Models

In this sub section, we will highlight some of the work that focuses on the EPQ model with production rate being treated as a decision variable. The classical economic production lot size model assumes a constant production rate that is predetermined and inflexible, and perfect quality. Recent models have removed the assumption of perfect quality while maintaining that of constant production rate. Machine production rate can easily be changed, observations indicate that as production rate is increased, tool costs increase. An example of drilling operation given by Conrad & McClamrock [1] can be quoted here, where a 10% change in production rate results in a 50% change in tool costs. Increasing the production rate also increases the probability of failure of the equipment.

Khouja and Mehrez [2] have extended the Economic Production Lot Size (EPLS) model to cases where the production rate is a decision variable. Unit production cost becomes a function of the production rate. Also the quality of the production process deteriorates with increased production rate. They have shown that for cases where increases in production rate lead to a significant deterioration in quality, the optimal production rate may be smaller than the rate that minimizes unit production cost and also that for cases where quality is largely independent of the production rate, the optimal production rate may be larger than the one that minimizes unit production cost.

Eiamkanchanalai and Benerjee [3] have developed a model for simultaneous determination of optimal run length and production rate for a single item under the assumption that production cost per unit is a quadratic function of production rate.

Larsen [4] has investigated the use of production rate as a decision variable and its adjustment when a change in demand is observed, (he has referred it as response mechanism). He has looked at the production rate as a mechanism to responding to changes in demand, and has shown that it might be optimal as a response to increased demand to decrease the production rate.

Al-Fawzan & Al-Sultan [5] have extended EPQ model for fixed interval demands by considering production rate to be controllable. They have developed mathematical models to minimize the total cost in two cases, i.e. with and without shortages and have found optimal production rates for both cases.

Eiamkanchanalai [6] has extended classical Economic Lot Scheduling Problem (ELSP) by treating the production rate of each item as decision variable and incorporating a linear penalty function associated with unused capacity where each item's production cost function is assumed to be strictly convex in its production rate.

2.2.2 Cutting Speed Models

Cutting speed has a direct effect on the production rate of the facility, so, if batches of multiple items are competing for the same facility in a machining system and the resulting initial schedule is infeasible then the cutting speeds can be modified in a way so that the corresponding production rates result in a feasible schedule. Numerous researchers have derived the optimal machining conditions but most of them have not taken into account lot size considerations.

Wysk et.al [7] introduce lot size considerations in determining the optimal cutting speed in a single item, single machine problem. Their approach is to set the cutting speed so that there is no unused portion of tool life in processing a lot. Their objective is to minimize processing time so they have not considered cost factors at all.

Goyal [8] has incorporated a dispatch policy in an optimal cutting speed model. He has developed a search procedure for determining the cutting speed and number of lots without any constraint on the available machine time.

Schweitzer and Seidmann [9] have considered tool economics in flexible manufacturing systems (FMS). They have taken processing rates as decision variables with the objective to find minimum cost processing rates given the FMS throughput target, the work-in-process (WIP) level, part routes, transporter delays and the variable capacity cost function for each machine.

Koulamas [10] has presented an iterative procedure for the simultaneous determination of the optimal cutting speed and lot size values in a machining system with the objective of minimizing the total cost consisting of machining and lot sizing costs. He has demonstrated that this scheme could provide a lower cost solution than separate procedures for determining cutting speed and lot size values.

2.2.3 Shelf Life Models

In general, products are stored in the inventory and consumed throughout the production period. Some products get spoiled if they are stored beyond certain period of time; this time limitation of life for a product is called *shelf life*. If the shelf life for the product is greater than the cycle time, it does not affect the inventory model; but for any of the products if the shelf life is less than the cycle time, it affects the whole inventory system, and the inventory model should be re- formulated accordingly. For an extensive review, the reader is referred to Goyal and Giri [11], Raafat [12] and Nahmias [13].

Silver [14] has considered a piece of manufacturing equipment dedicated to a family of items, under shelf life constraint. He has shown that, under the assumption that the change of production rate does not incur any additional cost, reducing production rate is more effective than the obvious choice i.e. reducing the cycle time.

Silver [15] has extended his original work [14] by allowing the individual item production rates to be controllable variables and has shown that under certain circumstances, significant potential cost savings can be realized by slowing down the production rate of just one key items in the family.

Sarker and Babu [16] have discussed a family of items of which only one item has a shelf life. They have concluded that the decision to either reduce the production rate(P) or the cycle time(T) depends mainly on production cost, holding cost and setup cost, with the production cost being the most significant.

Silver [17] has extended Sarker & Babu's work [16] by allowing all the items in the family to have shelf life. He has also considered the case where cost minimizing cycle length (T) results in violation of one of the shelf life constraint. He has proposed a procedure to simultaneously adjust both cycle length (T) and Production rate (P) and has shown that it is at least as good as adjusting either of these parameters alone.

Goyal [18] suggested an improvement in Sarker & Babu's solution [16] by allowing some items to be produced more than once in a cycle.

Viswanathan [19] pointed that Goyal's [18] suggestion can result in a lower cost solution but it does not guarantee a feasible production schedule because an important constraint in the original model [16] was that all items are produced with the same frequency.

Goyal [20] commented on Viswanathan's [19] note that the discrepancy in their notes could be attributed to the fact that the problem given in Sarker & Babu [16] is essentially an extension of the Economic Lot Scheduling Problem (ELSP). Therefore, any of the methods available to solve the ELSP can be used to solve this problem.

Viswanathan & Goyal [21] have suggested an algorithm to determine both the optimal production rate for all the items as well as the optimal cycle time for a situation where more than one item might have a binding shelf life constraint.

Viswanathan & Goyal [22] have provided an extension to their earlier work [21] by allowing planned back orders in order to lower the average inventory level, which in turn will make the shelf life constraint less restrictive.

2.2.4 Other Related Models

It is common to produce several items in a single high-capacity facility rather than to use a dedicated facility for each of the items. Typically these high capacity facilities are only capable of producing one item at a time (e.g. typical job shop production). After production run for an item is finished it may be necessary to prepare or setup the facility, at the cost of time or money, for the production of different item. For detailed review, the reader is referred to Elmaghraby [23], Dobson [24], Gallego [25] and Zipkin [26].

Under the assumption of constant demand and production rates, the economic lot scheduling problem (ELSP) is concerned with scheduling the production of the items in the facility to minimize the long-run average inventory carrying and setup costs. **Gallego** [27] has considered a revision of the ELSP where the item's production rates can be reduced from the current or nominal production rates resulting in substantial savings for facilities with low demand rates and/or setup times.

Here, we will briefly look at the models that can not be classified into the above classes yet they deal with variable production rates.

Arizono et. al. [28] have presented a procedure based on the minimal lattice paths for analyzing the probabilities that an overflow of inventory or a shortage occur in the controllable production system and the uncontrollable production-rates system. They have discussed the effects of varying production rates on inventory control by comparing these two systems.

Kok [29] has discussed production-inventory model with variable production rate for the case where the demand process is a compound poisson process, the production rate is controlled by (m, M) policy. He has derived accurate approximations for operating characteristics of the system and has shown that sequential determination of $M - m$ and m yields a rule that minimizes average costs subject to a service level constraint.

2.3 Problem Definition and Scope

The problem to be tackled is the Economic Production Quantity problem with variable production rate such a case may arise in the case of deteriorating process for single-machine, single-product case where the production rate changes from the initial level P_1 to a certain smaller level P_2 because of a shift in the deteriorating process with or without effect on quality.

Other applications could be for the case of n machines producing items and at some time t, one or more breaks down, effectively reducing the production rate, or if at time t there is a shift in the process and after this time certain amount of items produced are of sub-standard quality, this situation may also be represented by this variable production rate model. Another major area where this type of modeling can be helpful is quality problems, as one can consider the effective production rate (production rate corresponding to good quality items) changes when there is a shift in the production process, thereby reducing the production rate of the good quality items. For the purpose of this thesis, we will assume the production of single item on single machine without shelf life constraints.

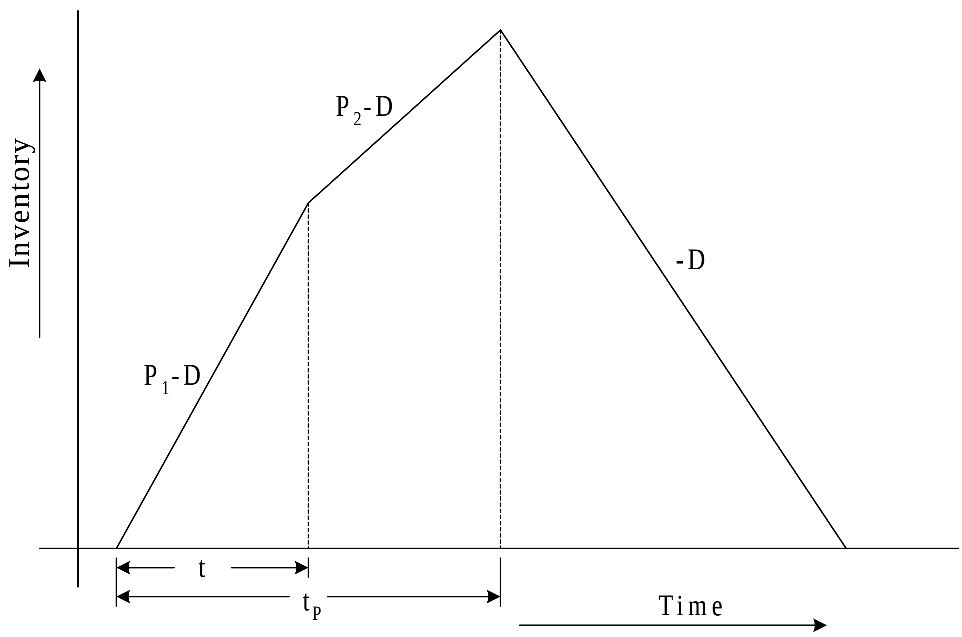

Figure 2.1: Inventory profile with varying production rate

As shown in Figure 2.1, the cycle starts with production rate P_1 and after a time t, which could be deterministic or random with a known distribution, the production rate shifts to a value P_2, such that $P_2 < P_1$, during the whole cycle the demand is assumed to be fixed and known, with $P_1 > P_2 > D$, the time period during which we produce is t_P, which is our decision variable, that is how long we should continue production so as to minimize the total inventory costs which is the sum of setup, production and holding costs.

2.4 Summary

In this chapter we have looked at the available literature in the area of production economics and classified the literature, briefly discussed the contribution of those authors. Later, we clearly presented the problem of variable production rate and briefly explained the scope of the problem.

Chapter 3

Variable Production Rate model with deterministic time to shift

3.1 Introduction

The purpose of this chapter is to develop a frame work for the whole thesis by presenting the basic model and analyzing it. We will consider time to shift from one to the other production rate as deterministic.

We will state assumption and notations that will be used throughout the chapter, develop the mathematical model explicitly stating different costs involved.

3.2 Assumptions

In developing the model the following assumptions were made:

1. The process begins with a production rate P_1.
2. After time t, which is known and deterministic the process shifts to a lower production rate P_2.
3. Demand is deterministic and constant with $P_1 \geq P_2 \geq D$.
4. Cycle is repeated after every T time units.
5. The unit production cost is assumed to be convex function of the production rate (the unit production cost first decreases with the increasing production rate until a minimum unit production cost is reached after which the unit production cost starts to increase with the production rate)
6. The items that are short (because of the shift) may or mat not be produced overtime at some extra cost.

3.3 Model Development

In this section, we will develop the mathematical cost model for deterministic time to shift.

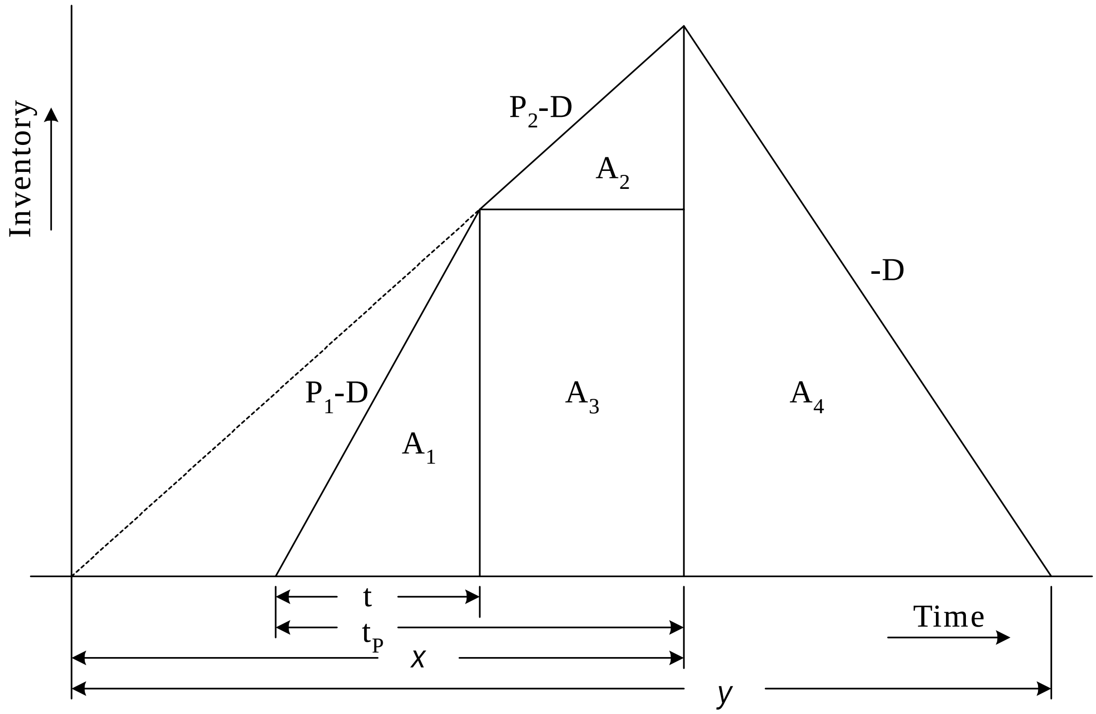

Figure 3.1: Inventory profile with varying production rate without overtime production

3.3.1 Setup Cost

Let the setup cost of the system which also includes restoration at the beginning of the cycle be S, so

$$\text{Setup Cost} = S \tag{3.1}$$

3.3.2 Production Cost

The unit production cost will be taken to have a convex shape, as different production rates result in different production costs. As the production rate is increased, some costs such as the labor and energy costs are spread over more units while per unit tool/die costs decrease. The net result is that unit production costs decreases until an ideal design production rate of the machine is reached, beyond which unit production cost increases. such as $aP + \frac{b}{P}$

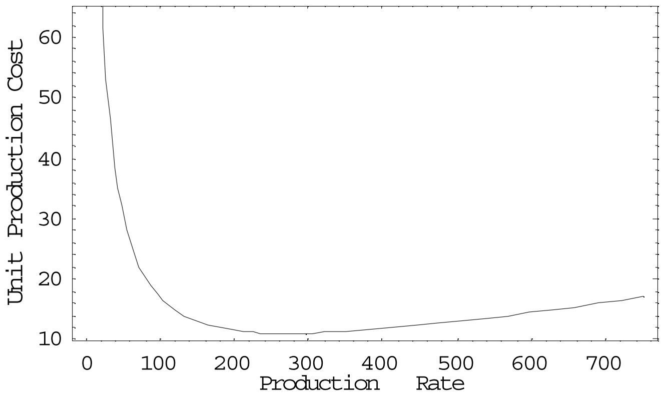

Figure 3.2: Plot showing variation of unit production cost against production rate for $a = 0.02$ and $b = 1500$

$$\text{Production Cost} = \left(aP_1 + \frac{b}{P_1}\right) P_1 t + \left(aP_2 + \frac{b}{P_2}\right) P_2 (t_P - t) \tag{3.2}$$

3.3.3 Holding Cost

In general, the average holding cost is given by

$$\text{Avg. Holding cost} = h \times \frac{\text{Area under the inventory curve}}{\text{Total cycle time}}$$

From figure (1),

$$\text{Avg. Holding cost} = h \times \frac{A_1 + A_2 + A_3 + A_4}{T} \tag{3.3}$$

$$A_1 = \frac{1}{2} t I_1$$

where $I_1 = (P_1 - D)t$, so

$$A_1 = \frac{t^2(P_1 - D)}{2}$$

Similarly,

$$A_2 = \frac{1}{2}(t_P - t)(I_2 - I_1)$$

where $I_2 - I_1 = (P_2 - D)(t_P - t)$, so

$$A_2 = \frac{(t_P - t)^2(P_2 - D)}{2}$$

$$\begin{aligned} A_3 &= I_1(t_P - t) \\ A_3 &= (P_1 - D)[tt_P - t^2] \end{aligned}$$

Following similar treatment,

$$\begin{aligned} A_4 &= \frac{1}{2}(y - x)I_2 \\ I_2 &= (P_2 - D)x \\ A_4 &= \frac{(P_2 - D)}{2}(yx - x^2) \end{aligned}$$

where $x = \frac{I_2}{(P_2 - D)}$, and $I_2 = (P_2 - D)(t_P - t) + (P_1 - D)t$

$$x = t_P + \frac{(P_1 - P_2)t}{(P_2 - D)}$$

now,

$$y = \frac{P_2}{D}x$$

So, by replacing x in y, we have

$$y = \frac{P_2}{D}\left\{t_P + \frac{(P_1 - P_2)t}{(P_2 - D)}\right\}$$

Replacing x and y in A_4 and simplifying, we have

$$A_4 = \frac{(P_2 - D)^2}{2D} t_P^2 + \frac{(P_2 - D)(P_1 - P_2)}{D} t_P t + \frac{(P_1 - P_2)^2}{2D} t^2$$

Replacing A_1, A_2, A_3 and A_4 in (3.3) and rearranging, we have

$$HC = h \left[t^2 \left\{ \frac{(P_1 - P_2)^2 - D(P_1 - P_2)}{2D} \right\} + t_P t \left\{ \frac{P_2(P_1 - P_2)}{D} \right\} + t_P^2 \left\{ \frac{P_2(P_2 - D)}{2D} \right\} \right] \tag{3.4}$$

3.3.4 Total Cost Equation

Total cost is given by

$$\begin{aligned} TC_1 &= S + \left(aP_1 + \frac{b}{P_1} \right) P_1 t + \left(aP_2 + \frac{b}{P_2} \right) P_2 (t_P - t) \\ &+ h \left[t^2 \left\{ \frac{(P_1 - P_2)^2 - D(P_1 - P_2)}{2D} \right\} + t_P t \left\{ \frac{P_2(P_1 - P_2)}{D} \right\} \right. \\ &+ \left. t_P^2 \left\{ \frac{P_2(P_2 - D)}{2D} \right\} \right] \end{aligned} \tag{3.5}$$

3.3.5 Cycle Time

From figure 3.1

$$CL_1 = t_P + (y - x)$$

Simplifying after substituting x and y in above, we have

$$CL_1 = \frac{P_2 t_P + (P_1 - P_2) t}{D} \tag{3.6}$$

3.3.6 Total Cost per unit time

Total total cost per unit time is given by:

$$\begin{aligned}
ETC(t_P) &= \left[S + \left(aP_1 + \frac{b}{P_1}\right) P_1 t + \left(aP_2 + \frac{b}{P_2}\right) P_2(t_P - t)\right. \\
&+ h\left\{t^2 \left(\frac{(P_1 - P_2)^2 - D(P_1 - P_2)}{2D}\right) + t_P t \left(\frac{P_2(P_1 - P_2)}{D}\right)\right. \\
&+ \left.\left. t_P^2 \left(\frac{P_2(P_2 - D)}{2D}\right)\right\}\right] \Big/ \left\{\frac{P_2 t_P + (P_1 - P_2)t}{D}\right\} \qquad (3.7)
\end{aligned}$$

3.3.7 Model Analysis

The decision variable for our problem is t_P. We will take partial derivative of the total cost equation and set it to zero to obtain the value of the production time t_P. i.e.,

$$\frac{dTC}{dt_P} = 0$$

After setting partial derivative with respect to t_P to zero, and solving for t_P we get

$$\begin{aligned}
t_P^* &= \left[Dht(P_1 - P_2) - htP_2(P_1 - P_2) \pm \sqrt{Dh}\sqrt{P_2 - D}\right. \\
&\left.\sqrt{2SP_2 + ht^2 P_1 P_2 - ht^2 P_1^2 - 2bt(P_1 - P_2) + 2atP_1 P_2(P_1 - P_2)}\right] \Big/ \\
&\{hP_2(P_2 - D)\}
\end{aligned}$$

the above value of t_P^* will minimize the total cost.

Only the positive sign in the latter part of the above function can give us a positive root because the first part has a negative sign therefore we will drop the negative sign from the latter part. We have now, after rearranging

$$t_P^* = \Big[-ht(P_1 - P_2)(P_2 - D) + \sqrt{Dh}\sqrt{P_2 - D} \sqrt{2SP_2 + ht^2P_1P_2 - ht^2P_1^2 - 2bt(P_1 - P_2) + 2atP_1P_2(P_1 - P_2)}\Big] \Big/ \{hP_2(P_2 - D)\} \tag{3.8}$$

For the root to be real and positive, the term under the root must be positive. For which the condition will be

$$ht^2\left[P_2\left\{P_1 - P_2 + D\right\}\right] + 2Dt\left(b - aP_1P_2\right) < \frac{2SDP_2}{P_1 - P_2}$$

3.3.8 Numerical Results

A numerical example along with sensitivity analysis will be presented to gauge the performance of the model. Also we will try to give intuitive explanation of the results obtained in this section. The data used for this numerical example is $P_1 = 270$ [1], $h = 2, a = 0.02, b = 1500, S = 370, t = 0.05, P_2 = 180$ and $D = 20$ unless otherwise stated.

Higher P_2 means higher production cost, the system will try and minimize this by decreasing t_P^*, as a direct consequence of increased production time, less holding cost has to be incurred and therefore the ETC decreases

Increasing time to shift means more time is available for higher production rate

[1] this value corresponds to minimum unit production cost, refer section 3.3.2

P_1/P_2	t_P^*	ETC*	CL
5	0.96979	664.70	3.158434
4	0.960362	582.93	3.747473
3	0.821093	502.29	4.144919
2	0.602778	428.36	4.406252
1.5	0.472537	398.88	4.477836

Table 3.1: Effect of changing P_1/P_2 on the Optimal parameters

t	t_P^*	$t_P^*(EPQ)$	ETC*	CL
0.005	0.50349		400.68	4.553906
0.05	0.472537		398.88	4.477836
0.1	0.437518		396.67	4.38766
0.2	0.365325		391.57	4.187929
0.638	8.84E-05	0.331104	384.66	2.871796

Table 3.2: Effect of changing t on the Optimal parameters

(having lower unit production cost), which tend to decrease the cycle length, the systems attempts to obtain optimal trade-off between setup cost on one hand (decreasing with increasing cycle length) and holding and production costs on the other hand (increasing with increasing cycle length).

P_2/D	t_P^*	ETC*	CL
5	0.678624	634.04	0.140725
3	0.970075	957.82	0.331692
2.5	1.124013	1,110.99	0.459605
2	1.382248	1,331.80	0.703624
1.5	1.965149	1,676.82	1.326766

Table 3.3: Effect of changing P_2/D on the Optimal parameters

Increasing D will try to force the cycle length to become smaller but as the cycle length decreases, the impact of fixed setup costs becomes more, so the system tries to obtain optimal trade-off between setup cost on one hand and the production and holding costs on the other, the result of which appears in higher production time and expected costs.

3.4 With overtime production

In this section, we will revise the model we have just developed to include the possibility of producing the difference overtime.

All the costs will remain the same, except that we will have an extra additional cost because of overtime production and the area under the inventory triangle will be that of Economic Production Quantity model.

3.4.1 Overtime production Cost

Overtime production costs results from producing the difference caused by the shift in the production rate, the decrease in the number of items produced by the decreased production rate will be produced using the facility overtime.

$$\text{Overtime production Cost} = \alpha(P_1 - P_2)(t_P - t) \tag{3.9}$$

3.4.2 Holding Cost

Holding cost will be the same as that of EPQ case.

$$\text{Holding cost} = h\left\{\frac{P_1(P_1 - D)t_P^2}{2D}\right\} \tag{3.10}$$

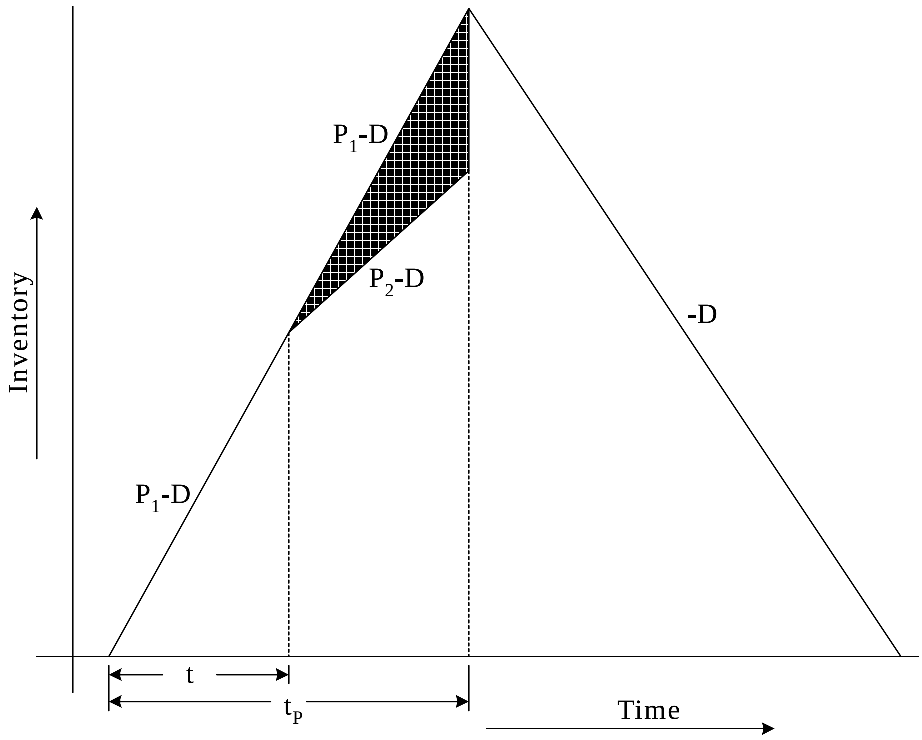

Figure 3.3: Inventory profile with varying production rate with overtime production

3.4.3 Total Cost Equation

So, the total cost equation in this case will be:

$$
\begin{aligned}
TC_2 &= S + \left(aP_1 + \frac{b}{P_1}\right) P_1 t + \left(aP_2 + \frac{b}{P_2}\right) P_2 (t_P - t) \\
&+ h\left\{\frac{P_1(P_1 - D)t_P^2}{2D}\right\} + \alpha(P_1 - P_2)(t_P - t)
\end{aligned} \tag{3.11}
$$

3.4.4 Cycle Length

As seen in figure 3.3, the cycle length will also be affected and will be reduced because of overtime production the difference, so, the cycle length will be the same as that in EPQ.

$$CL_2 = \frac{P_1}{D} t_P \tag{3.12}$$

3.4.5 Average Cost per unit time

$$\begin{aligned} TC &= \left[S + \left(aP_1 + \frac{b}{P_1} \right) P_1 t + \left(aP_2 + \frac{b}{P_2} \right) P_2 (t_P - t) \right. \\ &+ \left. h \left\{ \frac{P_1 (P_1 - D) t_P^2}{2D} \right\} + \alpha (P_1 - P_2)(t_P - t) \right] \Big/ \left\{ \frac{P_1}{D} t_P \right\} \end{aligned} \tag{3.13}$$

3.4.6 Model Analysis

We will take partial derivative of the total cost equation with respect to t_P and set it to zero to obtain the value of the production time t_P. i.e.,

$$\frac{dTC}{dt_P} = 0$$

$$t_P^* = \frac{\sqrt{2D}\sqrt{S + at(P_1^2 - P_2^2) - t\alpha(P_1 - P_2)}}{\sqrt{hP_1(P_1 - D)}} \tag{3.14}$$

the above value of t_P will minimize the total cost.

From the above equation t_P^* will be real and positive iff

$$t < \frac{S}{\alpha(P_1 - P_@) - a(P_1^2 - P_2^2)} \tag{3.15}$$

3.4.7 Numerical Result

Similar data as used in the previous section will be used here also: As we increase P2, the quantity that has to be produced overtime will decrease, so to minimize the

P_1/P_2	t_P^*	ETC*	CL
5	0.257615	564.24	3.477806
4	0.264348	550.03	3.568692
3	0.274766	527.16	3.70934
2	0.292967	484.59	3.955054
1.5	0.308161	446.52	4.160168

Table 3.4: Effect of changing P_1/P_2 on the Optimal parameters

expected cost per unit time t_P^* will increase (because of higher α). As the overtime production cost per unit increases, the quantity produced overtime will decrease and the cost per unit will also increase as a direct consequence.

t	t_P^*	$t_P^*(EPQ)$	ETC*	CL
0.005	0.328881		456.89	4.439899
0.05	0.308161		446.52	4.160168
0.1	0.283366		434.13	3.825441
0.15	0.256183		420.54	3.458468
0.3737	0.003311	0.3311	384.66	0.044699

Table 3.5: Effect of changing t on the Optimal parameters

Increasing time to shift means more time is available for higher production rate (having lower unit production cost), which tend to decrease the cycle length, the systems attempts to obtain optimal trade-off between setup cost on one hand (decreasing with increasing cycle length) and holding and production costs on the other hand (increasing with increasing cycle length). Increasing Demand will try to force the cycle length to become smaller but as the cycle length decreases, the impact of fixed setup costs becomes more, so the system tries to obtain optimal trade-off between

P2/D	tP*	ETC*	CL
5	0.427342	726.40	3.205064
3	0.582369	1,121.93	2.62066
2.5	0.657001	1,312.97	2.463752
2	0.770402	1,593.34	2.311205
1.5	0.974489	2,047.01	2.192601

Table 3.6: Effect of changing P_2/D on the Optimal parameters

setup cost on one hand and the production nd holding costs on the other.

3.5 Conclusions

In this chapter, we developed a mathematical model with and without overtime production for deterministic time to shift which involved setup cost, production cost, holding cost and overtime production cost (in the latter case), analyzed these models to obtain closed form solutions and obtained bounds on the value of t for t_P^* to be real and positive. We also presented some numerical results along with sensitivity analysis to highlight the model's performance.

Chapter 4

Variable Production Rate model with stochastic time to shift

4.1 Introduction

In this chapter, we will build on the frame work set in the last chapter and extend the model for stochastic time to shift. We will state assumption and notations that to be used later, develop the mathematical model, analyze the model and present some numerical examples along with sensitivity analysis to measure the performance of the model.

4.2 Assumptions

In developing the model the following assumptions were made:

1. The process begins with a production rate P_1.

2. After time t, which is a random variable following general distribution, the process shifts to a lower production rate P_2.

3. Demand is deterministic and constant with $P_1 \geq P_2 \geq D$.

4. Cycle is repeated after every T time units.

5. The unit production cost is assumed to be convex in the production rate (the unit production cost first decreases with the increasing production rate until a minimum unit production cost is reached after which the unit production cost starts to increase with the production rate)

4.3 Model Development

We will develop models for two kinds of production costs. In the first case we will consider convex unit production cost and in the other case we will consider production costs which are made up of two components namely production costs per unit per unit time which includes labor and energy costs and the other is the production costs per unit time which includes material costs etc.

4.3.1 Convex Unit Production Cost

There are two possible scenarios while dealing with the random time to shift, these are:

- When the shift occurs during the production run, i.e. $t \in [0, t_P]$
- When the shift occurs after the production run, i.e. $t \in [t_P, \infty)$

We will discuss each of these cases in detail in the following section.

4.3.2 When the shift occurs during production run

In this section, we will develop the mathematical model for the case where the shift occurs during the production run.

Setup Cost

Let the setup cost of the system which also includes restoration at the beginning of the cycle be S, so

$$\text{Setup Cost} = S \tag{4.1}$$

Production Cost

$$\text{Production Cost} = \left(aP_1 + \frac{b}{P_1}\right) P_1 t + \left(aP_2 + \frac{b}{P_2}\right) P_2 (t_P - t) \tag{4.2}$$

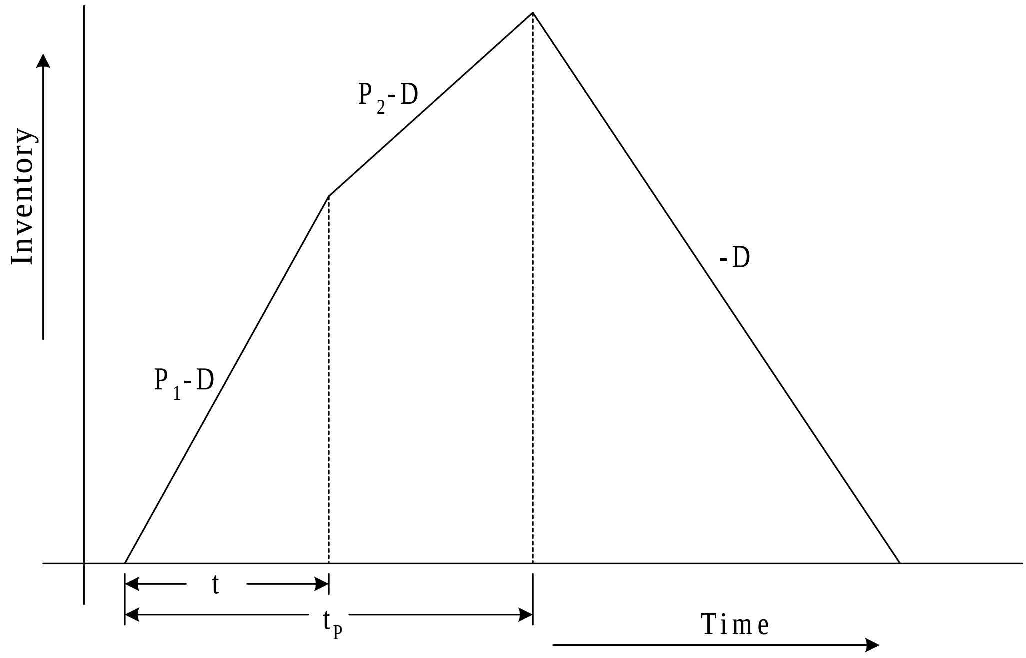

Figure 4.1: Inventory profile with varying production rate when $t \leq t_P$

Holding Cost

Referring to section 3.3.3, the holding cost is given by

$$HC = h\left[t^2\left\{\frac{(P_1-P_2)^2 - D(P_1-P_2)}{2D}\right\} + t_P t\left\{\frac{P_2(P_1-P_2)}{D}\right\} + t_P^2\left\{\frac{P_2(P_2-D)}{2D}\right\}\right] \tag{4.3}$$

Lost Production Cost

In many cases, there is a penalty associated in the event of producer's failure to provide the agreed quantity.

$$\text{Lost Production Cost} = C_l(P_1-P_2)(t_P-t) \tag{4.4}$$

Total Cost Equation

Total cost is given by

$$
\begin{aligned}
TC_1 &= S + \left(aP_1 + \frac{b}{P_1}\right) P_1 t + \left(aP_2 + \frac{b}{P_2}\right) P_2(t_P - t) \\
&+ h\left[t^2 \left\{\frac{(P_1 - P_2)^2 - D(P_1 - P_2)}{2D}\right\} + t_P t \left\{\frac{P_2(P_1 - P_2)}{D}\right\}\right. \\
&+ \left. t_P^2 \left\{\frac{P_2(P_2 - D)}{2D}\right\}\right] + C_l(P_1 - P_2)(t_P - t) \qquad (4.5)
\end{aligned}
$$

Cycle Time

From figure 4.1

$$CL_1 = t_P + (y - x)$$

Simplifying after substituting x and y in above, we have

$$CL_1 = \frac{P_2 t_P + (P_1 - P_2)t}{D} \qquad (4.6)$$

4.3.3 When the shift occurs after production

A similar approach to the one used in the last section will be adopted.

Setup Cost

Setup cost for the system is assumed to be fixed and is given by

$$\text{Setup Cost} = S \qquad (4.7)$$

Production Cost

The total production cost in this case will be

$$\begin{aligned} \text{Production Cost} &= \left(aP_1 + \frac{b}{P_1}\right) P_1 t_P \\ &= t_P \left(aP_1^2 + b\right) \end{aligned}$$

Holding Cost

From figure 4.2 the holding cost is given by

$$\text{Holding cost} = h(A_1 + A_2) \tag{4.8}$$

$$A_1 = \frac{1}{2}\, t_P I_1$$

where,

$$I_1 = (P_1 - D)\; t_P$$

so,

$$A_1 = \frac{1}{2}(P_1 - D)t_P^2$$

Similarly, $A_2 = \frac{1}{2}(y - x)I_1$, from the figure, we have

$$y - x = t_P \left[\frac{P_1}{D} - 1\right]$$

So,

$$A_2 = \frac{t_P^2 (P_1 - D)^2}{2D}$$

Replace A_1 and A_2 in (4.8), and simplifying, we have

$$\text{Holding cost} = h \left\{ \frac{P_1 (P_1 - D) t_P^2}{2D} \right\} \tag{4.9}$$

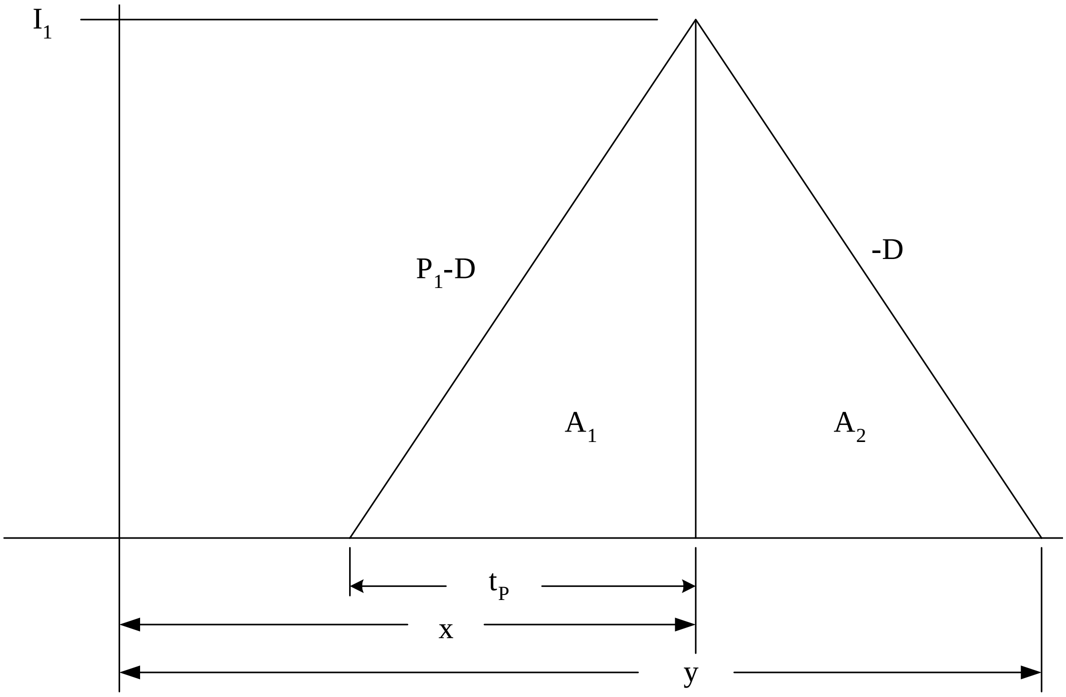

Figure 4.2: Inventory profile with varying production rate with $t > t_P$.

Total Cost Equation

Total cost is the sum of setup, production and holding costs.

$$TC_2 = S + t_P \left(aP_1^2 + b\right) + h \left\{ \frac{P_1(P_1 - D)t_P^2}{2D} \right\} \tag{4.10}$$

Cycle Length

From figure 4.3, we have

$$CL_2 = \frac{P_1}{D} t_P \tag{4.11}$$

4.3.4 Expected Total Cost per unit time

In this section we will develop the model for general distribution and relax the assumption of deterministic time to shift as was used in the earlier section i.e. $0 \leq t < \infty$. Expected cost per cycle will be

$$E[TC] = \int_0^{t_P} TC_1 f(t)dt + \int_{t_P}^{\infty} TC_2 f(t)dt$$

Where TC_1 and TC_2 are given by eqns. (4.5) and (4.10) respectively. Similarly Expected cycle length is given by

$$E[CL] = \int_0^{t_P} CL_1 f(t)dt + \int_{t_P}^{\infty} CL_2 f(t)dt$$

Where CL_1 and CL_2 are given by eqns. (4.6) and (4.11) respectively.

Using Renewal Reward Theorem, the expected cost per unit time $ETC(t_P)$ will be

$$ETC(t_P) = \frac{E[TC]}{E[CL]}$$

Upon substitution, we get

$$\begin{aligned}
ETC(t_P) &= \left\{ \int_0^{t_P} \left[S + \left(aP_1 + \frac{b}{P_1} \right) P_1 t + \left(aP_2 + \frac{b}{P_2} \right) P_2(t_P - t) \right.\right. \\
&+ h \left\{ t^2 \left(\frac{(P_1 - P_2)^2 - D(P_1 - P_2)}{2D} \right) + t_P t \left(\frac{P_2(P_1 - P_2)}{D} \right) \right. \\
&+ \left. t_P^2 \left(\frac{P_2(P_2 - D)}{2D} \right) \right\} + C_l(P_1 - P_2)(t_P - t) \Bigg] f(t)dt + \int_{t_P}^{\infty} \left[S + t_P \left(aP_1^2 + b \right) \right. \\
&+ \left. \left. h \left\{ \frac{P_1(P_1 - D)t_P^2}{2D} \right\} \right] f(t)dt \right\} \Big/ \left\{ \int_0^{t_P} \frac{P_2 t_P + (P_1 - P_2)t}{D} f(t)dt \right. \\
&+ \left. \int_{t_P}^{\infty} \frac{P_1}{D} t_P f(t)dt \right\} \qquad (4.12)
\end{aligned}$$

4.3.5 Exponential Time to shift

We will take an example of exponential time to shift, for which the probability density function is given by $f(t) = \lambda e^{-\lambda t}$. So Expected cost per cycle will be

$$E[TC] = \int_0^{t_P} TC_1 \lambda e^{-\lambda t} dt + \int_{t_P}^{\infty} TC_2 \lambda e^{-\lambda t} dt$$

where TC_1 and TC_2 are given by equations (4.5) and (4.10) respectively. Similarly, Expected cycle length for exponential time to shift is given by

$$E[CL] = \int_0^{t_P} CL_1 \lambda e^{-\lambda t} dt + \int_{t_P}^{\infty} CL_2 \lambda e^{-\lambda t} dt$$

where CL_1 and CL_2 are given by equations (4.6) and (4.11) respectively. So, the expected total cost per unit time after simplification would be

$$\begin{aligned} ETC(t_P) \;=\; & \left[S + \left\{ C_2 P_2 t_P + C_l(P_1 - P_2)t_P + h\frac{P_2(P_2 - D)t_P^2}{2D} \right\} (1 - e^{-\lambda t_P}) \right. \\ + \; & \left\{ C_1 P_1 - C_2 P_2 - C_l(P_1 - P_2) + \frac{hP_2(P_1 - P_2)}{D} t_P \right\} \left(\frac{1}{\lambda} - \frac{e^{-\lambda t_P}}{\lambda} - t_P e^{-\lambda t_P} \right) \\ + \; & \left\{ C_1 P_1 t_P + \frac{hP_1(P_1 - D)t_P^2}{2D} \right\} e^{-\lambda t_P} \\ + \; & \left. \frac{h\left\{(P_1 - P_2)^2 - D(P_1 - P_2)\right\}}{2D} \left\{ \frac{2}{\lambda^2} - \frac{e^{-\lambda t_P}\left(2 + 2\lambda t_P + \lambda^2 t_P^2\right)}{\lambda^2} \right\} \right] \Big/ \\ & \left[\frac{e^{-\lambda t_P} t_P P_1}{D} + \frac{\left(\frac{1}{\lambda} - \frac{e^{-\lambda t_P}}{\lambda} - t_P e^{-\lambda t_P}\right)(P_1 - P_2)}{D} + \frac{\left(1 - e^{-\lambda t_P}\right) t_P P_2}{D} \right] \end{aligned} \tag{4.13}$$

Where $C_1 = aP_1 + \frac{b}{P_1}$ and $C_2 = aP_2 + \frac{b}{P_2}$.

Model analysis

In this section we will discuss the properties of the model in detail and state theorems that we will use to derive the results.

Let

$$TC' = \frac{dTC}{dt_P}$$
$$CL' = \frac{dCL}{dt_P}$$

Note that

$$\frac{d}{dt_P}ETC = \frac{(CL)\ (TC') - (TC)\ (CL')}{(CL)^2}$$

Let

$$\psi(t_P) = (CL)\ (TC') - (TC)\ (CL') \tag{4.14}$$

Through simple manipulation, i.e., by simple substitution it can be shown that

$$\lim_{t_P \to 0} \psi(t_P) = -\frac{SP_1}{D}$$

For $\lim_{t_P \to \infty} \psi(t_P)$, the terms that have Exponential part will grow to zero as $t \to \infty$ So, we are left with only those terms that does not involve exponential part. i.e.

$$\begin{aligned} \lim_{t_P \to \infty} \psi(t_P) &= \frac{bP_1}{D\lambda} + \frac{C_l P_1^2}{D\lambda} - \frac{SP_2}{D} - \frac{bP_2}{D\lambda} + \frac{hP_1P_2}{D\lambda^2} - \frac{ht_P P_1 P_2}{D\lambda} - \frac{C_l P_1 P_2}{D\lambda} \\ &- \frac{aP_1^2 P_2}{D\lambda} - \frac{ht_P^2 P_2^2}{2D} - \frac{hP_2^2}{D\lambda^2} + \frac{ht_P P_2^2}{D\lambda} + \frac{aP_1 P_2^2}{D\lambda} + \frac{ht_P^2 P_2^3}{2D^2} \\ &+ \frac{ht_P P_1 P_2^2}{D^2\lambda} - \frac{ht_P P_2^3}{D^2\lambda} \end{aligned}$$

Rearranging, we have

$$\begin{aligned}\lim_{t_P\to\infty}\psi(t_P) &= \frac{bP_1}{D\lambda}+\frac{C_lP_1^2}{D\lambda}-\frac{SP_2}{D}-\frac{bP_2}{D\lambda}+\frac{hP_1P_2}{D\lambda^2}-\frac{C_lP_1P_2}{D\lambda}-\frac{aP_1^2P_2}{D\lambda}\\ &- \frac{hP_2^2}{D\lambda^2}+\frac{aP_1P_2^2}{D\lambda}+\frac{ht_PP_1P_2^2}{D^2\lambda}-\frac{ht_PP_2^3}{D^2\lambda}-\frac{ht_PP_1P_2}{D\lambda}+\frac{ht_PP_2^2}{D\lambda}\\ &+ \frac{ht_P^2P_2^3}{2D^2}-\frac{ht_P^2P_2^2}{2D}\end{aligned}$$

$$\begin{aligned}\lim_{t_P\to\infty}\psi(t_P) &= \frac{bP_1}{D\lambda}+\frac{C_lP_1^2}{D\lambda}-\frac{SP_2}{D}-\frac{bP_2}{D\lambda}+\frac{hP_1P_2}{D\lambda^2}-\frac{C_lP_1P_2}{D\lambda}-\frac{aP_1^2P_2}{D\lambda}\\ &- \frac{hP_2^2}{D\lambda^2}+\frac{aP_1P_2^2}{D\lambda}+t_P\left\{\frac{hP_1P_2^2}{D^2\lambda}-\frac{hP_2^3}{D^2\lambda}-\frac{hP_1P_2}{D\lambda}+\frac{hP_2^2}{D\lambda}\right\}\\ &+ t_P^2\left\{\frac{hP_2^3}{2D^2}-\frac{hP_2^2}{2D}\right\}\end{aligned}$$

After careful grouping, we have

$$\begin{aligned}\lim_{t_P\to\infty}\psi(t_P) &= \frac{bP_1}{D\lambda}+\frac{C_lP_1^2}{D\lambda}-\frac{SP_2}{D}-\frac{bP_2}{D\lambda}+\frac{hP_1P_2}{D\lambda^2}-\frac{C_lP_1P_2}{D\lambda}-\frac{aP_1^2P_2}{D\lambda}\\ &- \frac{hP_2^2}{D\lambda^2}+\frac{aP_1P_2^2}{D\lambda}+t_P(P_1-P_2)\left[\frac{hP_2}{D\lambda}\left\{\frac{P_2}{D}-1\right\}\right]\\ &+ t_P^2\frac{hP_2^2}{2D}\left\{\frac{P_2}{D}-1\right\}\end{aligned}$$

From our assumption, $P_1 \geq P_2 \geq D$, so $P_2/D \geq 1$, so as $t_P \to \infty$, $\psi(t_P) \to \infty$. That is

$$\lim_{t_P\to\infty}\psi(t_P)\to\infty$$

So, the function has at least one local minimum. To further emphasize our point, we would just show empirically that the function behaves in a convex manner by plotting Expected cost per unit time and time of production. We will show here that for an EPQ model, if $P_1 > P_2$, then $t^*_{P_1} < t^*_{P_2}$ and also that $ETC^*_{P_1} < ETC^*_{P_2}$.

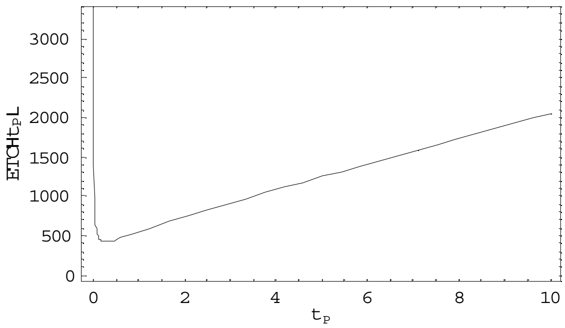

Figure 4.3: Plot of Expected Total Cost per unit time showing convex shape for convex unit production cost

The proof is as follows:

$$ETC = \frac{S + CPt_P + \frac{hP(P-D)}{2D}t_P^2}{\frac{P}{D}t_P}$$

Now, take partial derivative with respect to t_P and equate to zero and simplify, we get

$$t_P^* = \sqrt{\frac{2SD}{hP(P-D)}}$$

Now, for P_1 we will have $t_{P_1}^* = \sqrt{\frac{2SD}{hP_1(P_1-D)}}$ and similarly for P_2, we have $t_{P_2}^* = \sqrt{\frac{2SD}{hP_2(P_2-D)}}$. Also, by our assumption $P_1 > P_2 > D$, and S, D, h are the same, so we need to show that $P_1(P_1 - D) > P_2(P_2 - D)$, which is fairly simple since $P_1 > P_2$ so $P_1 - D > P_2 - D$, and by transitive property of in- equivalence, we know that

$$\text{if } a > b \ \& \ c > d \text{ then } ac > bd$$

So, we conclude that $t_{P_1}^* < t_{P_2}^*$.

Now, we will focus our attention to the Expected total cost per unit time. If we replace the above t_P^* in the EPQ cost model, we get

$$ETC^* = \sqrt{\frac{2SDh(P-D)}{P}} + CD$$

Now, if $P_1 > P_2$, then

$$ETC^*_{P_1} = \sqrt{\frac{2SDh(P_1-D)}{P_1}} + CD$$

and

$$ETC^*_{P_2} = \sqrt{\frac{2SDh(P_2-D)}{P_2}} + CD$$

Since $S, D\&h$ are constants, then all we need to show is that

$$\sqrt{1-\frac{D}{P_1}} < \sqrt{1-\frac{D}{P_2}}$$

By our assumption that $P_1 > P_2 > D$, we have $\frac{1}{P_1} < \frac{1}{P_2} < \frac{1}{D}$, or $\frac{D}{P_1} < \frac{D}{P_2} < 1$. So, we conclude that for an Economic Production Quantity model, if $P_1 > P_2$ then $t^*_{P_1} < t^*_{P_2}$ and $ETC^*_{P_1} < ETC^*_{P_2}$.

Numerical Results

A numerical example along with sensitivity analysis will be presented to gauge the performance of the model. We will use "The Golden Section Rule" to solve the problem. The data used for this numerical example is $P_1 = 270, P_2 = 180, D = 20, S = 370, h = 2, a = 0.02, b = 1500, C_l = 20$ and $\lambda = 10$ unless otherwise stated. The reason for keeping relatively high value of λ is that if it is kept smaller, the probability of having a shift is small and the difference between regular EPQ model

λ	t_P^*	ETC^*	Setup	Prod.	Hold.	Lost Prod.
0.1	0.3269	387.15	84.29	219.32	81.36	2.17
1.0	0.3064	407.8	93.79	221.01	73.60	19.39
5.0	0.2654	474.31	123.32	225.85	56.21	68.93
10.0	0.27805	524.27	128.49	230.09	53.35	112.34
20.0	0.36975	565.41	105.44	234.91	63.51	161.55

Table 4.1: Effect of changing λ on the Optimal parameters

and the one presented in this chapter is not evident.

As we increase λ, the mean time to shift decreases and less time is available for P_1, but as the per unit production cost is lower for P_1 than P_2, so logically, the time of production should decrease to decrease the overall cost, but as λ is increased further, the portion of time during which P_1 is available becomes even smaller, in order to meet the demand the production time t_P starts to increase. All the time increase of λ will prompt an increase in Expected total cost per unit time because of higher unit production cost of P_2.

P_1/P_2	t_P^*	ETC^*	Setup	Prod.	Hold.	Lost Prod.
5	0.11218	830.08	358.82	261.06	22.36	187.48
4	0.120011	796.31	332.55	257.48	23.57	182.47
3	0.13752	739.99	294.05	250.92	25.71	169.41
2	0.1912	628.56	198.3	241.01	35.86	153.36
1.5	0.27805	524.27	128.49	230.09	53.35	112.34

Table 4.2: Effect of changing P_1/P_2 on the Optimal parameters

Increasing P_2 decreases the unit production cost as unit production cost is convex in nature and is minimum for $P = 270$ therefore reducing the Expected total cost per unit time and higher production times.

P_2/D	t_P^*	ETC^*	Setup	Prod.	Hold.	Lost Prod.
5	0.3522	891.81	186.65	416.81	61.1	229.24
3	0.45475	1432.57	244.6	698.73	65.84	423.4
2.5	0.5131	1699.89	266.59	840.3	66.3	526.7
2	0.60798	2097.62	281.21	1053.98	67.17	695.26
1.5	0.84809	2751.40	274.66	1412.41	64.7	999.61

Table 4.3: Effect of changing P_2/D on the Optimal parameters

Increasing demand will try to force the cycle length to become smaller but as the cycle length decreases, the impact of fixed setup costs becomes more, so the system tries to obtain optimal trade-off between setup cost on one hand and the production and holding costs on the other.

4.3.6 Two stage production cost

Here, we will consider production costs which are made up of two components namely production costs per unit per unit time which includes labor and energy costs and the other is the production costs per unit time which includes material costs etc. The only change in the model in this case will be in the production cost.

Production Cost

The production cost per unit item for the cycle will be

$$C_m P_1 t + C_m P_2 (t_P - t)$$

where as that per unit time will be

$$C_p t + C_p(t_P - t)$$

or $C_p t_P$, but this term does not have to be divided by the cycle time as it is already cost per unit time. So, re-writing the model, we replace P_1 by P and P_2 by βP. For general distribution whose probability density function is given by $f(t)$, the Expected total cost per unit time will be

$$\begin{aligned}
ETC(t_P) &= \Bigg\{ \int_0^{t_P} \Bigg[S + C_m P t + C_m \beta P (t_P - t) \\
&+ h\Bigg\{ t^2 \left(\frac{P^2(1-\beta)^2 - DP(1-\beta)}{2D} \right) + t_P t \left(\frac{\beta P^2 (1-\beta)}{D} \right) \\
&+ t_P^2 \left(\frac{\beta P(\beta P - D)}{2D} \right) \Bigg\} + C_l P(1-\beta)(t_P - t) \Bigg] f(t)dt + \int_{t_P}^{\infty} \Bigg[S \\
&+ C_m P t_P + h \left\{ \frac{P(P-D)t_P^2}{2D} \right\} \Bigg] f(t)dt \Bigg\} \Bigg/ \Bigg\{ \int_0^{t_P} \frac{\beta P t_P + P(1-\beta)t}{D} f(t)dt \\
&+ \int_{t_P}^{\infty} \frac{P}{D} t_P f(t)dt \Bigg\} + C_p t_P
\end{aligned} \tag{4.15}$$

4.3.7 Exponential Time to shift

We will take an example of exponential time to shift, for which the probability density function is given by $f(t) = \lambda e^{-\lambda t}$. So Expected cost per cycle will be

$$E[TC] = \int_0^{t_P} TC_1 \lambda e^{-\lambda t} dt + \int_{t_P}^{\infty} TC_2 \lambda e^{-\lambda t} dt$$

Where

$$
\begin{aligned}
TC_1 &= \Bigg[S + C_m Pt + C_m \beta P(t_P - t) + C_l P(1-\beta)(t_P - t) \\
&+ h\Bigg\{t^2\left(\frac{P^2(1-\beta)^2 - DP(1-\beta)}{2D}\right) + t_P t\left(\frac{\beta P^2(1-\beta)}{D}\right) \\
&+ t_P^2\left(\frac{\beta P(\beta P - D)}{2D}\right)\Bigg\}\Bigg]
\end{aligned}
$$

and

$$
TC_2 = \left[S + C_m P t_P + h\left\{\frac{P(P-D)t_P^2}{2D}\right\}\right]
$$

Similarly, Expected cycle length for exponential time to shift is given by

$$
E[CL] = \int_0^{t_P} CL_1 \lambda e^{-\lambda t} dt + \int_{t_P}^{\infty} CL_2 \lambda e^{-\lambda t} dt
$$

Where

$$
CL_1 = \frac{\beta P t_P + P(1-\beta)t}{D}
$$

and

$$
CL_2 = \frac{P}{D} t_P
$$

The expected total cost per unit time would be

$$
ETC(t_P) = \frac{E[TC]}{E[CL]} + C_p t_P
$$

Upon simplification, we have

$$
\begin{aligned}
ETC(t_P) &= \left[S + \left\{\beta C_m P t_P + C_l P(1-\beta)t_P + h\frac{\beta P(\beta P - D)}{2D}t_P^2\right\}(1-e^{-\lambda t_P})\right. \\
&+ \left\{C_m P(1-\beta) - C_l P(1-\beta) + h\frac{\beta P^2(1-\beta)}{D}t_P\right\}\left(\frac{1}{\lambda} - \frac{e^{-\lambda t_P}}{\lambda} - t_P e^{-\lambda t_P}\right) \\
&+ \frac{h\left\{P^2(1-\beta)^2 - DP(1-\beta)\right\}}{2D}\left\{\frac{2}{\lambda^2} - \frac{e^{-\lambda t_P}\left(2 + 2\lambda t_P + \lambda^2 t_P^2\right)}{\lambda^2}\right\} \\
&+ \left.\left\{C_m P t_P + h\frac{P(P-D)}{2D}t_P^2\right\}e^{-\lambda t_P}\right] \Big/ \left[\frac{e^{-\lambda t_P} t_P P}{D} + \frac{\left(1-e^{-\lambda t_P}\right)t_P \beta P}{D}\right. \\
&+ \left.\frac{\left(\frac{1}{\lambda} - \frac{e^{-\lambda t_P}}{\lambda} - t_P e^{-\lambda t_P}\right)P(1-\beta)}{D}\right] + C_p t_P \qquad (4.16)
\end{aligned}
$$

Model analysis

The above equation can be modified slightly to be written in the standard fraction form of $ETC = \frac{TC}{CL}$, then the notation $\psi(t_P)$ used in 4.3.5 can be used here. Now, through simple substitution it can be shown that

$$\lim_{t_P \to 0} \psi(t_P) = -\frac{SP}{D}$$

For $\lim_{t_P \to \infty} \psi(t_P)$, we know that as $t \to \infty$, $e^{-t} \to 0$ So, we are left with only those terms that does not involve exponential part. i.e.

$$
\begin{aligned}
\lim_{t_P \to \infty} \psi(t_P) &= -\frac{PS\beta}{D} - \frac{ht_P^2 P^2 \beta^2}{2D} + \frac{ht_P^2 P^3 \beta^3}{2D^2} + \frac{hP^2\beta}{D\lambda^2} - \frac{hP^2\beta^2}{D\lambda^2} - \frac{ht_P P^2 \beta}{D\lambda} \\
&+ \frac{ht_P P^2 \beta^2}{D\lambda} + \frac{ht_P P^3 \beta^2}{D^2\lambda} - \frac{ht_P P^3 \beta^3}{D^2\lambda} + \frac{P^2 C_l}{D\lambda} - \frac{P^2 \beta C_l}{D\lambda} + \frac{t_P^2 P^2 \beta^2 C_p}{D^2} \\
&+ \frac{P^2 C_p}{D^2\lambda^2} - \frac{2P^2\beta C_p}{D^2\lambda^2} + \frac{P^2\beta^2 C_p}{D^2\lambda^2} + \frac{2t_P P^2 \beta C_p}{D^2\lambda} - \frac{2t_P P^2 \beta^2 C_p}{D^2\lambda}
\end{aligned}
$$

Rearranging, we have

$$\begin{aligned}\lim_{t_P\to\infty}\psi(t_P) &= -\frac{PS\beta}{D}+\frac{P^2C_l}{D\lambda}-\frac{P^2\beta C_l}{D\lambda}+\frac{hP^2\beta}{D\lambda^2}-\frac{hP^2\beta^2}{D\lambda^2}+\frac{P^2C_p}{D^2\lambda^2}-\frac{2P^2\beta C_p}{D^2\lambda^2}\\ &+ \frac{P^2\beta^2C_p}{D^2\lambda^2}+\frac{ht_PP^3\beta^2}{D^2\lambda}-\frac{ht_PP^3\beta^3}{D^2\lambda}+\frac{2t_PP^2\beta C_p}{D^2\lambda}-\frac{2t_PP^2\beta^2C_p}{D^2\lambda}\\ &- \frac{ht_PP^2\beta}{D\lambda}+\frac{ht_PP^2\beta^2}{D\lambda}+\frac{t_P^2P^2\beta^2C_p}{D^2}-\frac{ht_P^2P^2\beta^2}{2D}+\frac{ht_P^2P^3\beta^3}{2D^2}\end{aligned}$$

$$\begin{aligned}\lim_{t_P\to\infty}\psi(t_P) &= -\frac{PS\beta}{D}+\frac{P^2C_l}{D\lambda}-\frac{P^2\beta C_l}{D\lambda}+\frac{hP^2\beta}{D\lambda^2}-\frac{hP^2\beta^2}{D\lambda^2}+\frac{P^2C_p}{D^2\lambda^2}-\frac{2P^2\beta C_p}{D^2\lambda^2}\\ &+ \frac{P^2\beta^2C_p}{D^2\lambda^2}+t_P\left\{\frac{hP^3\beta^2}{D^2\lambda}-\frac{hP^3\beta^3}{D^2\lambda}+\frac{2P^2\beta C_p}{D^2\lambda}-\frac{2P^2\beta^2C_p}{D^2\lambda}\right.\\ &- \left.\frac{hP^2\beta}{D\lambda}+\frac{hP^2\beta^2}{D\lambda}\right\}+t_P^2\left\{\frac{P^2\beta^2C_p}{D^2}-\frac{hP^2\beta^2}{2D}+\frac{hP^3\beta^3}{2D^2}\right\}\end{aligned}$$

After careful grouping, we have

$$\begin{aligned}\lim_{t_P\to\infty}\psi(t_P) &= -\frac{PS\beta}{D}+\frac{P^2C_l}{D\lambda}-\frac{P^2\beta C_l}{D\lambda}+\frac{hP^2\beta}{D\lambda^2}-\frac{hP^2\beta^2}{D\lambda^2}+\frac{P^2C_p}{D^2\lambda^2}-\frac{2P^2\beta C_p}{D^2\lambda^2}\\ &+ \frac{P^2\beta^2C_p}{D^2\lambda^2}+t_P(1-\beta)\left[\frac{hP^2\beta}{D\lambda}\left\{\frac{P\beta}{D}-1\right\}+\frac{2P^2\beta C_p}{D^2\lambda}\right]\\ &+ t_P^2\left[\frac{hP^2\beta^2}{2D}\left\{\frac{P\beta}{D}-1\right\}+\frac{P^2\beta^2C_p}{D^2}\right]\end{aligned}$$

From our assumption, $P_1 \geq P_2 \Rightarrow P \geq \beta P$, or $0 \leq \beta \leq 1$ and as $P_2 \geq D$ so $\beta P \geq D$ as $t_P \to \infty$, $\psi(t_P) \to \infty$. That is

$$\lim_{t_P\to\infty}\psi(t_P) \to \infty$$

So, the function has at least one local minimum and also empirically the function behaves in a convex manner.

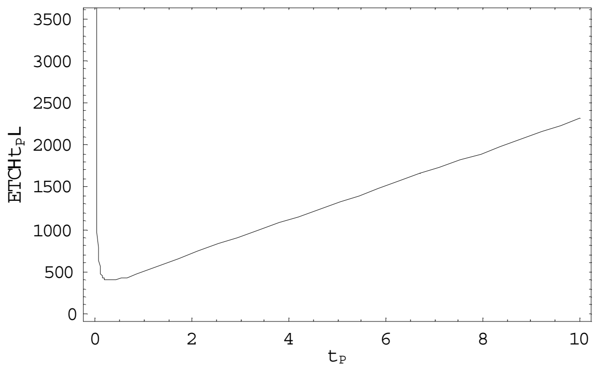

Figure 4.4: Plot of Expected Total Cost per unit time showing convex shape for two stage production cost

Numerical Result

Here, we will present a numerical example along with sensitivity analysis to show the performance of the model. We will use "The Golden Section Rule" to solve the problem. The data used for this numerical example is $P = 270, \beta = \frac{2}{3}, D = 20, S = 370, h = 2, C_m = 7, C_p = 25, C_l = 20$ and $\lambda = 10$ unless otherwise stated.

λ	t_P^*	ETC^*	Setup	Prod.	Hold.	Lost Prod.	$C_p t_P$
0.1	0.31423	315.85	87.68	140	78.22	2.08	7.86
1.0	0.29372	334.29	97.65	140	70.67	18.63	7.34
5.0	0.26054	394.98	123.32	140	56.21	68.93	6.51
10.0	0.27321	441.02	128.49	140	53.35	112.34	6.83
20.0	0.35224	479.55	108.98	140	61.48	160.27	8.81

Table 4.4: Effect of changing λ on the Optimal parameters

Increasing λ, decreases mean time to shift and less time is available for higher production rate P, and will result in a larger production loss, for which there is a much higher penalty associated than the production cost resulting in higher costs and lower production time but increasing λ further will result in increasing of t_P because then the share of P_1 is further reduced and in order to satisfy demand the production time must increase. Also, this will cause an increase in the cost because of the penalty associated with the lost production.

β	t_P^*	ETC^*	Setup	Prod.	Hold.	Lost Prod.	$C_p t_P$
$\frac{1}{5}$	0.1201	710.29	342.98	140	23.53	200.74	3
$\frac{1}{4}$	0.1297	680.44	323.6	140	24.29	189.67	3.12
$\frac{1}{3}$	0.15019	630.58	275.36	140	27.52	184.13	3.63
$\frac{1}{2}$	0.20387	531.99	192.08	140	36.99	157.97	4.97
$\frac{2}{3}$	0.27321	441.02	128.49	140	53.35	112.34	6.83

Table 4.5: Effect of changing β on the Optimal parameters

Increasing β will not have any direct effect on unit production cost as it is linear, if β is more then the lost production is less and hence lower penalty has to be paid resulting in lower expected cost.

$\beta P/D$	t_P^*	ETC^*	Setup	Prod.	Hold.	Lost Prod.	$C_p t_P$
5	0.35708	735.79	184.65	252	61.1	229.24	8.81
3	0.4644	1165.17	242.26	420	66.43	425	11.49
2.5	0.52113	1376.16	259.26	504	68.03	531.84	13.03
2	0.62065	1688.82	275.89	630	68.33	699.08	15.52
1.5	0.8324	2200.16	281.08	840	63.46	994.94	20.69

Table 4.6: Effect of changing $\beta P/D$ on the Optimal parameters

Increasing demand will try to force the cycle length to become smaller but as the cycle length decreases, the impact of fixed setup costs becomes more, so the system tries to obtain optimal trade-off between setup cost on one hand and the production, holding and penalty costs on the other.

4.4 Conclusions

In this chapter, we developed a mathematical model for random time to shift for two different types of production costs, the model involved setup, production, holding and lost production costs, analyzed these models to show that there will be at least one local minima, plotted the general shape of the cost function which suggests that this function may well be convex or quasi-convex. Later, we applied "Golden Section Rule" to solve the model numerically for the case of exponential time to shift.

Chapter 5

Stochastic time to shift with overtime

5.1 Introduction

In most practical situations, producer can not take the risk of running short, because depending on the specific contracts there could be severe penalties associated with short supply. These clauses of the contract sometimes force the producer to supply the items either by running overtime or out-sourcing from a third-party supplier but not both. In this chapter, we will extend the work in the previous chapter by adding overtime compensation of the short items.

5.2 Assumptions

In developing the model the following assumptions were made:

1. The process begins with a production rate P_1.

2. After time t, which is a random variable following general distribution, the process shifts to a lower production rate P_2.

3. Demand is deterministic and constant with $P_1 \geq P_2 \geq D$.

4. Cycle is repeated after every T time units.

5. The unit production cost is assumed to be convex in the production rate (the unit production cost first decreases with the increasing production rate until a minimum unit production cost is reached after which the unit production cost starts to increase with the production rate)

6. The items that are short (because of the shift) will be produced using over time shifts at some extra cost i.e. γ times the regular unit production cost, where $\gamma > 1$.

5.3 Convex unit production cost

In this section, we will develop the mathematical model which inherits all the characteristics of the models developed in the previous chapter and also provides the option of running overtime to compensate for the items short. The costs involved are the setup cost, production cost, holding cost and the overtime production cost.

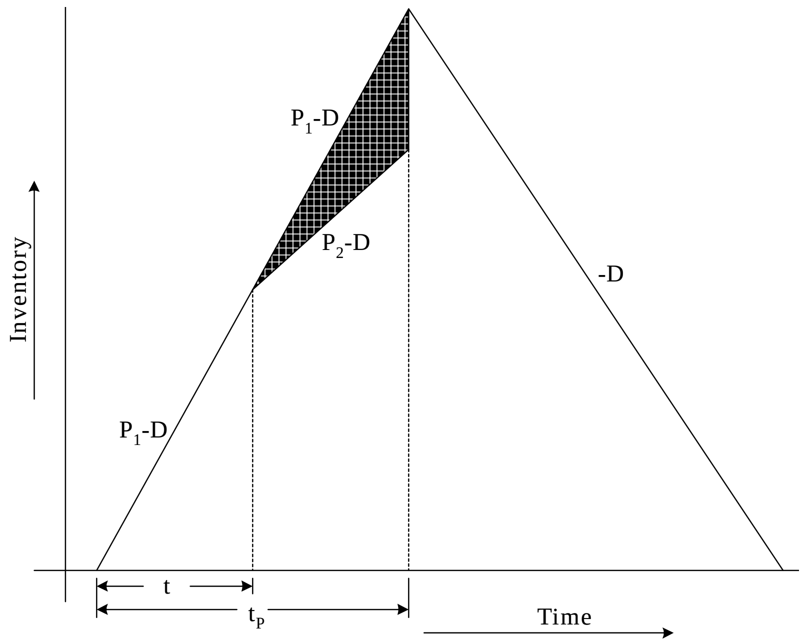

Figure 5.1: Inventory profile with varying production rate showing overtime production

5.3.1 When shift occurs during production

Setup Cost

Let the setup cost of the system which also includes restoration at the beginning of the cycle be S, so

$$\text{Setup Cost} = S \tag{5.1}$$

Production Cost

The unit production cost will be taken to have a convex shape, which leaves us with

$$\text{Production Cost} = \left(aP_1 + \frac{b}{P_1}\right) P_1 t + \left(aP_2 + \frac{b}{P_2}\right) P_2(t_P - t) \tag{5.2}$$

Overtime Production Cost

Referring to figure 5.1, and our assumption of convex unit production cost leads to the following cost equation

$$\text{Overtime Production Cost} = \gamma \left\{ aP_1 + \frac{b}{P_1} \right\} (P_1 - P_2)(t_P - t) \tag{5.3}$$

Holding Cost

Since we are running the facility overtime to produce the items short, so the holding cost will be the same as that incurred in the regular EPQ model [1].

$$\text{Holding cost} = h \left\{ \frac{P_1(P_1 - D)t_P^2}{2D} \right\} \tag{5.4}$$

Total Cost

The total cost per cycle will be given by

$$\begin{aligned} TC_1 &= S + \left(aP_1 + \frac{b}{P_1}\right) P_1 t + \left(aP_2 + \frac{b}{P_2}\right) P_2(t_P - t) + \gamma \left\{ aP_1 + \frac{b}{P_1} \right\} \\ &\quad (P_1 - P_2)(t_P - t) + h \left\{ \frac{P_1(P_1 - D)t_P^2}{2D} \right\} \end{aligned} \tag{5.5}$$

[1] for details refer to section 4.3.3

Cycle Length

The cycle length will be the same as

$$CL_1 = \frac{P_1}{D} t_P \tag{5.6}$$

5.3.2 When shift occurs after production

Both the cost and the cycle length in this case will be exactly the same as the one given in section 4.3.3, reproducing here, we have

$$TC_2 = S + t_P \left(aP_1^2 + b \right) + h \left\{ \frac{P_1(P_1 - D)t_P^2}{2D} \right\} \tag{5.7}$$

and

$$CL_2 = \frac{P_1}{D} t_P \tag{5.8}$$

Expected Cost per unit time

The total expected cost per cycle will be,

$$E[TC] = \int_0^{t_P} TC_1 f(t) dt + \int_{t_P}^{\infty} TC_2 f(t) dt$$

where TC_1 and TC_2 are given by equations (5.5) and (5.7), Similarly, Expected Cycle length is given by

$$E[TC] = \int_0^{t_P} CL_1 f(t) dt + \int_{t_P}^{\infty} CL_2 f(t) dt$$

where CL_1 and CL_2 are given by equations (5.6) and (5.8)

Using renewal reward theorem, the expected cost per unit time is

$$ETC(t_P) = \frac{E[TC]}{E[CL]}$$

5.4 Exponential time to shift

To better understand the model, let's take an example of exponential time to shift, for which the probability density function is given by $f(t) = \lambda e^{-\lambda t}$. So Expected cost per cycle will be

$$E[TC] = \int_0^{t_P} TC_1 \lambda e^{-\lambda t} dt + \int_{t_P}^{\infty} TC_2 \lambda e^{-\lambda t} dt$$

Similarly, Expected cycle length for exponential time to shift is given by

$$E[CL] = \int_0^{t_P} CL_1 \lambda e^{-\lambda t} dt + \int_{t_P}^{\infty} CL_2 \lambda e^{-\lambda t} dt$$

Upon simplification we will have

$$\begin{aligned} \mathrm{ETC}(t_P) &= \Bigg[S + \{C_1 P_1 - C_2 P_2 - \gamma C_1 (P_1 - P_2)\} \left\{ \frac{1}{\lambda} - \frac{e^{-\lambda t_P}}{\lambda} - t_P e^{-\lambda t_P} \right\} \\ &+ \{C_2 P_2 t_P + \gamma C_1 (P_1 - P_2) t_P\} \left(1 - e^{-\lambda t_P}\right) + C_1 P_1 t_P e^{-\lambda t_P} \\ &+ \frac{h P_1 (P_1 - D)}{2D} t_P^2 \Bigg] \Big/ \left\{ \frac{P_1 t_P}{D} \right\} \end{aligned} \tag{5.9}$$

Where $C_1 = aP_1 + \frac{b}{P_1}$ and $C_2 = aP_2 + \frac{b}{P_2}$.

5.4.1 Model analysis

Using the same definition as used in section 4.3.5, we have

$$\lim_{t_P \to 0} \psi(t_P) = -\frac{SP_1}{D}$$

For $\lim_{t_P \to \infty} \psi(t_P)$, the terms that have Exponential part will grow to zero as $t \to \infty$

So, we are left with only those terms that does not involve exponential part. i.e.

$$\begin{aligned} \lim_{t_P \to \infty} \psi(t_P) &= -\frac{SP_1}{D} - \frac{ht_P^2 P_1^2}{2D} + \frac{ht_P^2 P_1^3}{2D^2} + \frac{b\gamma P_1}{D\lambda} - \frac{aP_1^3}{D\lambda} + \frac{a\gamma P_1^3}{D\lambda} \\ &- \frac{b\gamma P_2}{D\lambda} - \frac{a\gamma P_1^2 P_2}{D\lambda} + \frac{aP_1 P_2^2}{D\lambda} \end{aligned}$$

After rearrangement, we have

$$\begin{aligned} \lim_{t_P \to \infty} \psi(t_P) &= -\frac{SP_1}{D} - \frac{a\gamma P_1^2 P_2}{D\lambda} + \frac{aP_1 P_2^2}{D\lambda} + \frac{b\gamma P_1}{D\lambda} - \frac{aP_1^3}{D\lambda} + \frac{a\gamma P_1^3}{D\lambda} \\ &- \frac{b\gamma P_2}{D\lambda} - \frac{ht_P^2 P_1^2}{2D} + \frac{ht_P^2 P_1^3}{2D^2} \end{aligned}$$

Re-grouping the last two terms, we have

$$\begin{aligned} \lim_{t_P \to \infty} \psi(t_P) &= -\frac{SP_1}{D} - \frac{a\gamma P_1^2 P_2}{D\lambda} + \frac{aP_1 P_2^2}{D\lambda} + \frac{b\gamma P_1}{D\lambda} - \frac{aP_1^3}{D\lambda} + \frac{a\gamma P_1^3}{D\lambda} \\ &- \frac{b\gamma P_2}{D\lambda} + t_P^2 \left(\frac{hP_1^2}{2D}\right) \left\{\frac{P_1}{D} - 1\right\} \end{aligned}$$

From, our assumption, $P_1 \geq P_2 \geq D$, so as $t_P \to \infty$, $\psi(t_P) \to \infty$. That is

$$\lim_{t_P \to \infty} \psi(t_P) \to \infty$$

Empirically, the function behaves in a convex manner,as shown in figure 5.4.1.

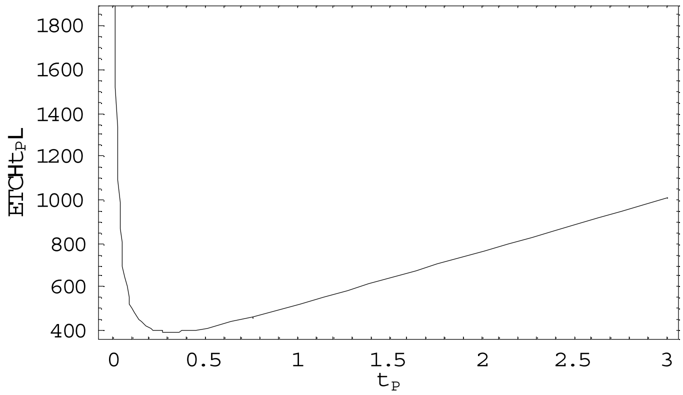

Figure 5.2: Plot of Expected Total Cost per unit time showing convex shape for convex unit production cost with over time

5.4.2 Numerical Results

A numerical example along with sensitivity analysis will be presented to gauge the performance of the model. We will use "The Golden Section Rule" to solve the problem. The data used for this numerical example is $P_1 = 270, P_2 = 180, D = 20, S = 370, h = 2, a = 0.02, b = 1500, \gamma = 1.3$ and $\lambda = 10$ unless otherwise stated.

λ	t_P^*	ETC^*	Setup	Prod.	Holding	Overtime Prod.
0.1	0.33173	385.23	83.84	218.14	81.72	1.54
1.0	0.3239	389.79	85.9	210.48	79.77	13.66
5.0	0.31124	402.21	89.45	189.81	76.6	46.36
10.0	0.31423	409.21	87.22	177.38	78.56	66.04
20.0	0.31907	414.26	85.9	168.5	79.77	80.09

Table 5.1: Effect of changing λ on the Model performance

As we increase the rate i.e. λ, the mean time to shift decreases, that is more quantity has to be produced after the shift, also increasing the over time production, for which the unit production cost is higher as $\gamma > 1$, but further increasing λ will mean that the impact of setup cost will be higher and hence an optimal trade-off is sought between setup cost and production, holding and overtime costs.

P_1/P_2	t_P^*	ETC^*	Setup	Prod.	Holding	Overtime Prod.
5	0.27022	469.41	103.28	151.75	66.34	148.05
4	0.27805	461.66	100.32	152.51	68.3	140.53
3	0.28588	449.41	95.87	154.77	71.47	127.28
2	0.29857	427.44	91.79	163.87	74.64	97.13
1.5	0.31423	409.21	87.22	177.38	78.56	66.04

Table 5.2: Effect of changing P_1/P_2 on the Optimal parameters

Keeping P_1 fixed, if we increase P_2, then less quantity has to be produced using overtime resulting in lower cost.

P_2/D	t_P^*	ETC^*	Setup	Prod.	Holding	Overtime Prod.
5	0.4313	658.23	115.69	311.37	99.78	131.39
3	0.5875	1007.72	139.96	507.89	123.37	236.49
2.5	0.6616	1175.76	149.12	605.4	131	290.22
2	0.7739	1421.62	159.36	750.87	139.3	372.08
1.5	0.9807	1817.76	168.5	991.55	146.39	511.32

Table 5.3: Effect of changing P_2/D on the Optimal parameters

Increasing the demand forces the cycle length to become smaller but as the cycle length decreases, the impact of fixed setup costs becomes more, so the system tries to obtain optimal trade-off between setup cost on one hand and the production, holding and overtime costs on the other.

As we increase the overtime unit production cost, the optimal values will tend to decrease the production time as much as possible, but the higher overtime costs will

γ	t_P^*	ETC^*	Setup	Prod.	Holding	Overtime Prod.
1.25	0.31423	406.66	87.22	177.38	78.56	63.5
1.50	0.3064	419.31	89.45	177.78	76.6	75.47
2.0	0.29072	444.14	95.87	178.9	71.47	97.91
5.0	0.19604	579.36	143.34	185.85	47.8	202.42
10.0	0.12485	757.48	219.52	193.38	31.21	313.23

Table 5.4: Effect of changing γ on the Optimal parameters

tend to push the costs upwards which was expected.

5.5 Two stage production cost

As in the previous section, here, we will develop a mathematical model for the case of two stage production cost with possibility of over time production and later compare the models with overtime and with lost production i.e. no overtime.

5.5.1 Two stage production cost

The production cost per unit item for the cycle will be

$$C_m P_1 t + C_m P_2 (t_P - t)$$

where as that per unit per unit time will be

$$C_p t + C_p (t_P - t)$$

or $C_p t_P$, but this term does not have to be divided by the cycle time as it is already cost per unit time. So, re-writing the model, we replace P_1 by P and P_2 by βP.

For general distribution whose probability density function is given by $f(t)$, the Expected total cost per unit time will be

$$
\begin{aligned}
ETC(t_P) &= \left\{ \int_0^{t_P} \left[S + C_m Pt + C_m \beta P(t_P - t) \right.\right. \\
&+ \left. h\left\{ \frac{P(P-D)t_P^2}{2D} \right\} + \gamma C_m P(t_P - t) \right] f(t)dt + \int_{t_P}^{\infty} \left[S \right. \\
&+ \left.\left. C_m P t_P + h\left\{ \frac{P(P-D)t_P^2}{2D} \right\} \right] f(t)dt \right\} \Big/ \left\{ \int_0^{t_P} \frac{P}{D} t_P f(t)dt \right. \\
&+ \left. \int_{t_P}^{\infty} \frac{P}{D} t_P f(t)dt \right\} + C_p t_P \qquad (5.10)
\end{aligned}
$$

5.5.2 Exponential time to shift

For exponential time to shift, the probability density function is given by $f(t) = \lambda\ e^{-\lambda t}$, replacing this in equation 5.10, we end up with

$$
\begin{aligned}
ETC(t_P) &= \left[S + \{C_m P(1-\beta) - \gamma C_m P\} \left\{ \frac{1}{\lambda} - \frac{e^{-\lambda t_P}}{\lambda} - t_P e^{-\lambda t_P} \right\} \right. \\
&+ \left. \{C_m P t_P(\beta + \gamma)\} \left\{1 - e^{-\lambda t_P}\right\} + \frac{hP(P-D)}{2D} t_P^2 + C_m P t_P e^{-\lambda t_P} \right] \Big/ \\
&\left\{ \frac{P}{D} t_P \right\} + C_p t_P \qquad (5.11)
\end{aligned}
$$

5.5.3 Model Analysis

If we take the limit of the Cost function as $t_P \to 0$, we have

$$\lim_{t_P \to 0} \psi(t_P) = -\frac{SP}{D}$$

For $\lim_{t_P \to \infty} \psi(t_P)$, we know that as $t \to \infty$, $e^{-t} \to 0$ So, we are left with only those terms that does not involve exponential part. i.e.

$$\lim_{t_P \to \infty} \psi(t_P) = -\frac{PS}{D} - \frac{P^2 C_m}{D\lambda} + \frac{P^2 \beta C_m}{D\lambda} + \frac{P^2 \gamma C_m}{D\lambda} + \frac{hP^3}{2D^2} t_P^2 - \frac{hP^2}{2D} t_P^2$$

After rearranging, we have

$$\lim_{t_P \to \infty} \psi(t_P) = -\frac{PS}{D} - \frac{P^2 C_m}{D\lambda} + \frac{P^2 \beta C_m}{D\lambda} + \frac{P^2 \gamma C_m}{D\lambda} + t_P^2 \left[\frac{hP^2}{2D} \left\{ \frac{P}{D} - 1 \right\} \right]$$

From, our assumption, $P_1 \geq P_2 \geq D$, so as $t_P \to \infty$, $\psi(t_P) \to \infty$. That is

$$\lim_{t_P \to \infty} \psi(t_P) \to \infty$$

Also the empirical behaviour of the function is convex, as shown in figure 5.3.

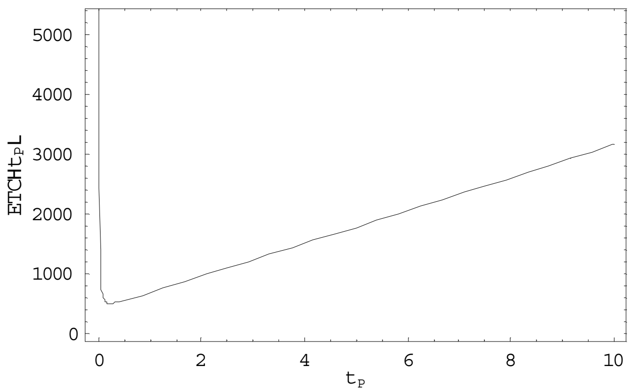

Figure 5.3: Plot of Expected Total Cost per unit time showing convex shape for 2 stage production cost with over time

5.5.4 Numerical Results

A numerical example along with sensitivity analysis will be presented to gauge the performance of the model. We will use "The Golden Section Rule" to solve the problem. The data used for this numerical example is $P = 270, \beta = \frac{2}{3}, D = 20, S = 370, h = 2, C_m = 7, C_p = 25, \gamma = 1.3$ and $\lambda = 10$ unless otherwise stated.

λ	t_P^*	ETC^*	Setup	Prod.	Hold.	O/T prod.	$C_p t_P$
0.1	0.31423	315.76	87.22	139.27	78.56	2.83	7.86
1.0	0.29071	332.11	95.87	133.92	71.47	23.7	7.15
5.0	0.25271	376.46	108.45	119.83	63.18	78.67	6.32
10.0	0.252708	404.01	108.45	110.33	63.18	115.73	6.32
20.0	0.27805	426.12	100.32	101.84	68.3	148.83	6.83

Table 5.5: Effect of changing λ on the Model performance

As we increase the rate i.e. λ, the mean time to shift decreases, that is more quantity has to be produced after the shift, also increasing the over time production, for which the unit production cost is higher as $\gamma > 1$, but further increasing λ will mean that t_P must increase so that the demand is satisfied although because of higher overtime unit production costs, total cost will be higher.

β	t_P^*	ETC^*	Setup	Prod.	Hold.	O/T Prod.	$C_p t_P$
$\frac{1}{5}$	0.28289	361.40	98.57	65.78	69.51	120.6	6.95
$\frac{1}{4}$	0.28289	366.06	98.57	70.42	69.51	120.6	6.95
$\frac{1}{3}$	0.27321	373.78	100.32	78.61	68.3	119.72	6.83
$\frac{1}{2}$	0.26538	389.02	105.19	94.88	65.13	117.31	6.51
$\frac{2}{3}$	0.25271	404.01	108.45	110.32	63.18	115.73	6.32

Table 5.6: Effect of changing β on the Optimal parameters

Increasing β will not have any direct effect on unit production cost as it is linear, as β increases the demand is met earlier therefore decreasing the production time. In our case increasing β does not decrease unit production cost, resulting in increase in the production cost. As there is a fixed per time production cost, the model will try and minimize this cost by decreasing production time, the overall effect of production cost upon increasing the β is more and hence increased cost.

$\beta P/D$	t_P^*	ETC^*	Setup	Prod.	Hold.	O/T Prod.	$C_p t_P$
5	0.33174	659.46	148.71	192.4	77.63	232.43	8.29
3	0.43425	1025.12	189.34	311.82	91.19	421.9	10.86
2	0.48497	1203.12	205.5	370.7	95.06	519.86	12
2.5	0.55195	1465.69	220.57	457.42	100.65	673.07	13.98
1.5	0.69185	1893.51	239.36	600.71	103.05	933.21	17.17

Table 5.7: Effect of changing $\beta P/D$ on the Optimal parameters

As we increase the demand keeping other parameters fixed, the cycle length would become smaller, this in turn increases the impact of fixed costs and therefore the system attempts to obtain optimal trade-off between the costs.

γ	t_P^*	ETC^*	Setup	Prod.	Hold.	O/T Prod.	$C_p t_P$
1.25	0.25755	399.54	108.45	110.32	63.18	111.28	6.32
1.50	0.24487	421.65	114.18	111.01	60.01	130.45	6
2	0.21654	464.05	129.46	112.72	52.93	163.66	5.29
5	0.12485	674.14	219.52	119.99	31.21	300.2	3.12
10	0.079005	926.6	346.9	125.59	19.75	432.16	1.97

Table 5.8: Effect of changing γ on the Optimal parameters

As we increase the overtime unit production cost, the optimal values will tend to decrease the production time as much as possible, but the higher overtime costs will tend to push the costs upwards which was expected.

5.6 Comparison: Lost production and overtime

Referring to the results of the numerical examples presented in this chapter and the preceding chapter, the important question that arises is

> "Whether to produce by using the machinery overtime or bear the penalty of the lost production."

Before answering this question, we will emphasize that this will vary depending on the specific contract clause, e.g. in some contracts the penalty for not being able to deliver the products on-time is high as in the case of items that have seasonal demands, in those cases it may be worth while to run the facility overtime but in other cases, the manufacturer either do not suffer any penalty or it is very small making the overtime a less fertile approach.

Having said that, we will emphasize that it is very easy to think that "It all depends on the per unit overtime cost and the per unit lost production penalty", we will point out the fact that it is not the case, because in case of lost production, the items that have to be held in the inventory are less than in the overtime case.

5.6.1 Convex unit production cost

Increasing $C_o = \gamma C_1$ forces the $t^*_{P_o}$ to decrease in order to minimize cost for that C_o, because of high overtime unit production cost the ETC increases also. In comparison for same C_l/C_o, the overtime option performs better than the lost production case because of smaller cycle length as it is not dependent on λ, as we are using relatively higher value of λ, which forces the cycle length to decrease and hence the impact of similar costs like the setup cost and the production cost are more than that in the overtime case where cycle length is independent of the shift.

$\frac{C_l}{C_o}$	$\frac{t^*_{P_l}}{t^*_{P_o}}$	$\frac{ETC^*_l}{ETC^*_o}$
10	0.77867	1.493
2	0.8504	1.3464
$\frac{4}{3}$	0.8933	1.2703
1	0.9466	1.2037
$\frac{2}{3}$	1.0672	1.093
0.5	1.197	1.005
0.1	3.3162	0.5359

Table 5.9: Comparative performance of the models with overtime and lost production for convex unit production cost

5.6.2 Two stage production cost

Increasing C_o, forces the production time to decrease, so as to limit the increase in cost, decreasing the cycle length along with it, which forces the impact of fixed costs to become higher and so can not be reduced indefinitely after the shift, which results in higher cost. For the same C_l and C_o, the lost production case performs slightly better because of the higher holding costs involved in the overtime case.

$\frac{C_l}{C_o}$	$\frac{t^*_{P_l}}{t^*_{P_o}}$	$\frac{ETC^*_l}{ETC^*_o}$
10	0.85267	1.427
2	1.1157	1.0617
$\frac{4}{3}$	1.32075	0.927
1	1.5303	0.8301
$\frac{2}{3}$	1.9867	0.7002
0.5	4.31324	0.4021
0.1	6.3768	0.28826

Table 5.10: Comparative performance of the models with overtime and lost production for two stage production cost

5.7 Conclusions

In this chapter, we basically extended the random time to shift model developed in the last chapter to incorporate the possibility of running the facility overtime for both types of production costs. We supported our discussion by using mathematical analysis tools such as the limit theorem to show the presence of at least one minima and empirically demonstrated the convex nature of the cost function. In the last section, we compared the findings of the model with overtime to that of lost production and found different results for different production costs.

Chapter 6

Stochastic time to shift with Restoration

6.1 Introduction

In most practical situations, whenever a shift occurs that reduces the production rate (such a shift could be belt slippage, tool wearing etc.) the possibility of inspection and restoring the system exists, we intend to model this situation in this chapter. The purpose of this chapter is to develop a model to jointly determine optimal production time and number of inspections. In this chapter, we build on the foundations developed in the last chapters and add to them restoration, obviously to determine whether restoration is needed or not, inspections also have to be performed. We will present a numerical example with sensitivity analysis at the end to study the performance of the model.

6.2 Assumptions

In developing the model the following assumptions were made:

1. The process begins with a production rate P_1.

2. After time t, which is a random variable following general distribution, the process shifts to a lower production rate P_2 and also after the shift a certain fraction α of the total components produced are of sub-standard quality.

3. Demand is deterministic and constant with $P_1 \geq P_2 \geq D$.

4. Inspections are performed at time periods where integrated hazard rates are equal.

5. Cycle is repeated after every T time units.

6. The unit production cost is assumed to be convex in production rate.

7. After a certain time h_j, where $j = 1, 2, \ldots n$ an inspection is done to determine whether the process is producing at full capacity or not, if the process is found to be operating at P_2, then an instantaneous restoration is done to bring the system back to P_1. (In most production scenarios save for high tech power

plants it is not often possible to know whether the shift has occurred unless the process in inspected)

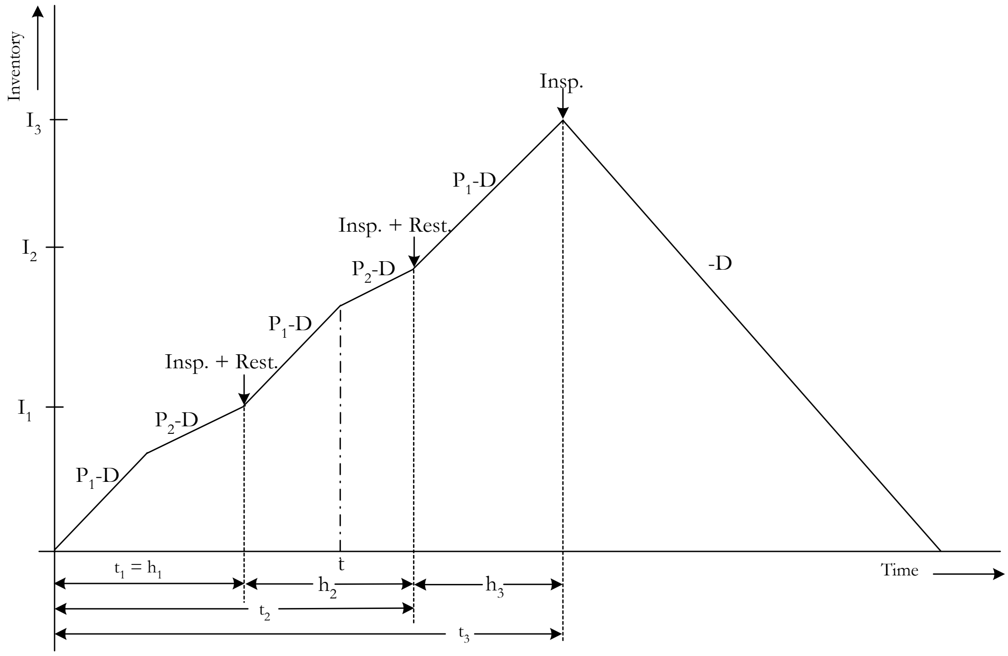

Figure 6.1: Inventory profile with varying production rate and restoration for $n = 3$

6.3 Model Development

6.3.1 Setup Cost

$$\text{Setup Cost} = S \tag{6.1}$$

6.3.2 Inspection Cost

$$\text{Inspection Cost} = nv \tag{6.2}$$

6.3.3 Production Cost

Referring to figure 6.1, and our assumption of convex unit production cost leads to the following cost equation

$$\begin{aligned}\text{Production Cost} &= \sum_{j=1}^{n} \left[\frac{\int_{t_{j-1}}^{t_j} \{C_1P_1(t - t_{j-1}) + C_2P_2(t_j - t)\}\, f(t)\, dt}{\overline{F}(t_{j-1})} \right. \\ &+ \left. \frac{\int_{t_j}^{\infty} C_1P_1(t_j - t_{j-1}) f(t)\, dt}{\overline{F}(t_{j-1})} \right] \qquad (6.3)\end{aligned}$$

Where $C_1 = aP_1 + \frac{b}{P_1}$ and $C_2 = aP_2 + \frac{b}{P_2}$.

6.3.4 Holding Cost

From figure 6.1 , the area under the inventory curve is given by

$$\sum_{j=1}^{n} \left\{ \frac{(I_{j-1} + I)}{2}(t - t_{j-1}) \;+\; \frac{(I + I_j)}{2}(t_j - t) \right\} \;+\; \frac{I_n^2}{2D}$$

so, the holding cost will be,

$$\begin{aligned}\text{H C} &= \sum_{j=1}^{n} \left[\left(\frac{h}{2}\right) \frac{\int_{t_{j-1}}^{t_j} [(I_{j-1} + I)\,(t - t_{j-1}) \;+\; (I + I_j)\,(t_j - t)]\, f(t)\, dt}{\overline{F}(t_{j-1})} \right. \\ &+ \left. \left(\frac{h}{2}\right) \frac{\int_{t_j}^{\infty} (I_j + I_{j-1})(t_j - t_{j-1}) f(t)\, dt}{\overline{F}(t_{j-1})} \right] + \frac{hI_n^2}{2D} \qquad (6.4)\end{aligned}$$

where I is the inventory level at the time to shift t. It is given by

$$\begin{aligned}I_j &= I_{j-1} + \frac{\int_{t_{j-1}}^{t_j} [(P_1 - D)(t - t_{j-1}) \;+\; (P_2 - D)(t_j - t)]\, f(t)\, dt}{\overline{F}(t_{j-1})} \\ &+ \frac{\int_{t_{j-1}}^{t_j} (P_1 - D)(t_j - t_{j-1}) f(t)\, dt}{\overline{F}(t_{j-1})} \qquad (6.5)\end{aligned}$$

The above equation (6.5) is significant as it provides a relationship between subsequent Inventory levels, also $I_0 = 0$.

6.3.5 Restoration Cost

It is the cost incurred in bringing the production rate back to the original value, i.e. P_1. Restoration cost has two components, fixed and time dependent. The time dependent part increases with the time that the system spends after the shift until being diagnosed.

$$\text{Restoration Cost} = \sum_{j=1}^{n} \left[\frac{\int_{t_{j-1}}^{t_j} [R_0 + R_1 (t_j - t)] \, f(t) \, dt}{\overline{F}(t_{j-1})} \right] \tag{6.6}$$

6.3.6 Lost Production Cost

The cost incurred if the producer fails to deliver the promised quantity of goods.

$$\text{Lost Production Cost} = \sum_{j=1}^{n} \left[\frac{\int_{t_{j-1}}^{t_j} C_l \, (P_1 - P_2) \, (t_j - t) f(t) \, dt}{\overline{F}(t_{j-1})} \right] \tag{6.7}$$

6.3.7 Quality Cost

The cost incurred if the goods produced are of sub-standard quality.

$$\text{Quality Cost} = \sum_{j=1}^{n} \left[\frac{\int_{t_{j-1}}^{t_j} \pi \alpha \, P_2 \, (t_j - t) f(t) \, dt}{\overline{F}(t_{j-1})} \right] \tag{6.8}$$

The total expected cost per cycle is then

$$\begin{aligned} E[TC] &= \text{Setup cost} + \text{Inspection cost} + \text{Production cost} + \text{Holding cost} \\ &+ \text{Restoration cost} + \text{Lost production cost} + \text{Quality cost} \end{aligned}$$

Substituting values in above equation, we have

$$
\begin{aligned}
E[TC] &= S + nv + \sum_{j=1}^{n} \left[\frac{\int_{t_{j-1}}^{t_j} \{C_1 P_1 (t - t_{j-1}) + C_2 P_2 (t_j - t)\} f(t)\, dt}{\overline{F}(t_{j-1})} \right. \\
&+ \left. \frac{\int_{t_j}^{\infty} C_1 P_1 (t_j - t_{j-1}) f(t)\, dt}{\overline{F}(t_{j-1})} \right] \\
&+ \sum_{j=1}^{n} \left[\left(\frac{h}{2}\right) \frac{\int_{t_{j-1}}^{t_j} [(I_{j-1} + I)\,(t - t_{j-1}) \; + \; (I + I_j)\,(t_j - t)]\, f(t)\, dt}{\overline{F}(t_{j-1})} \right. \\
&+ \left. \left(\frac{h}{2}\right) \frac{\int_{t_j}^{\infty} (I_j + I_{j-1})(t_j - t_{j-1}) f(t)\, dt}{\overline{F}(t_{j-1})} \right] + \frac{h I_n^2}{2D} \\
&+ \sum_{j=1}^{n} \left[\frac{\int_{t_{j-1}}^{t_j} [R_0 \; + R_1\,(t_j - t)]\, f(t)\, dt}{\overline{F}(t_{j-1})} \right] + \sum_{j=1}^{n} \left[\frac{\int_{t_{j-1}}^{t_j} \pi \alpha\, P_2\,(t_j - t) f(t)\, dt}{\overline{F}(t_{j-1})} \right] \\
&+ \sum_{j=1}^{n} \left[\frac{\int_{t_{j-1}}^{t_j} C_l\,(P_1 - P_2)\,(t_j - t) f(t)\, dt}{\overline{F}(t_{j-1})} \right] \qquad (6.9)
\end{aligned}
$$

6.3.8 Cycle Length

Refer fig 6.1 , the cycle length is given by

$$E[CL] = t_n + \frac{I_n}{D} \qquad (6.10)$$

where $t_n = t_{n-1} + h_n$. All the h_j's are related by the equal integrated hazard rate criterion [30], which states that the intervals are such that the integral of the hazard rate is always equal, i.e.,

$$\int_0^{t_1} r(t)\, dt = \int_{t_1}^{t_2} r(t)\, dt = \ldots = \int_{t_{n-1}}^{t_n} r(t)\, dt \qquad (6.11)$$

6.3.9 Expected Total Cost per unit time

Using the renewal reward theorem, Expected total cost per unit time or ETC is given by

$$ETC = \frac{E[TC]}{E[CL]}$$

Where $E[TC]$ and $E[CL]$ are given by equations (6.9) and (6.10) respectively.

6.4 Weibull Distributed time to shift

In this section, we will present a numerical example using Weibull distributed time to shift, for which probability density function is given by

$$f(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} e^{-(t/\theta)^{\beta}}$$

6.4.1 Interval Estimation

Using equal integrated hazard rate, as for weibull distribution, hazard rate is given by

$$r(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}$$

Consecutively solving equation (6.11) above and generalizing [30], we end up with

$$h_j = \left[j^{1/\beta} - (j-1)^{1/\beta}\right] h_1 \tag{6.12}$$

and then $t_j = t_{j-1} + h_j$, where $h_0 = t_0 = 0$.

6.4.2 Inventory Position

$$
\begin{aligned}
I_j &= I_{j-1} + \frac{\int_{t_{j-1}}^{t_j} [(P_1 - D)(t - t_{j-1}) + (P_2 - D)(t_j - t)]\, f(t)\, dt}{\overline{F}(t_{j-1})} \\
&+ \frac{\int_{t_j}^{\infty} (P_1 - D)(t_j - t_{j-1}) f(t)\, dt}{\overline{F}(t_{j-1})}
\end{aligned}
$$

$$
\begin{aligned}
I_j &= I_{j-1} + \frac{\int_{t_{j-1}}^{t_j} [(P_1 - D)(t - t_{j-1}) + (P_2 - D)(t_j - t)]\, \frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{\beta-1} e^{-(t/\theta)^\beta}\, dt}{\overline{F}(t_{j-1})} \\
&+ \frac{\int_{t_j}^{\infty} (P_1 - D)(t_j - t_{j-1}) \frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{\beta-1} e^{-(t/\theta)^\beta}\, dt}{\overline{F}(t_{j-1})}
\end{aligned}
$$

Solving above integral using the substitution $u = \left(\frac{t}{\theta}\right)^\beta$, we end up with

$$
\begin{aligned}
I_j &= I_{j-1} + \frac{(P_1 - P_2)\theta\, X_j - (P_1 - D)\, t_{j-1}\, Z_j + (P_2 - D)\, t_j\, Z_j}{W_{j-1}} \\
&+ \frac{W_j (P_1 - D)(t_j - t_{j-1})}{W_{j-1}} \qquad (6.13)
\end{aligned}
$$

Where X_j, Y_j, Z_j and W_j are given by

$$
X_j = \Gamma \left\{ 1 + \frac{1}{\beta}, \left(\frac{t_{j-1}}{\theta}\right)^\beta, \left(\frac{t_j}{\theta}\right)^\beta \right\} \qquad (6.14)
$$

$$
Y_j = \Gamma \left\{ 1 + \frac{2}{\beta}, \left(\frac{t_{j-1}}{\theta}\right)^\beta, \left(\frac{t_j}{\theta}\right)^\beta \right\} \qquad (6.15)
$$

$$
Z_j = e^{-(t_{j-1}/\theta)^\beta} - e^{-(t_j/\theta)^\beta} \qquad (6.16)
$$

$$
W_j = e^{-(t_j/\theta)^\beta} \qquad (6.17)
$$

where $\Gamma \{a, b, c\}$ is given by

$$
\Gamma \{a, b, c\} = \int_b^c x^{a-1} e^{-x}\, dx
$$

6.4.3 Production cost

Using similar procedure as described in section 6.4.2, we have

$$\mathrm{PC} = \sum_{j=1}^{n} \frac{[\theta X_j(C_1P_1 - C_2P_2) - (C_1P_1t_{j-1} - C_2P_2t_j)Z_j + C_1P_1(t_j - t_{j-1})W_j]}{W_{j-1}} \tag{6.18}$$

6.4.4 Holding Cost

$$\begin{aligned} \mathrm{HC} &= \sum_{j=1}^{n} \left[\left(\frac{h}{2}\right) \left\{ 2I_{j-1}\left\{\theta X_j - t_{j-1}Z_j\right\} + (P_1 - D)\left\{\theta^2 Y_j - 2t_{j-1}\theta X_j + t_{j-1}^2 Z_j\right\} \right.\right. \\ &+ 2I_j\left\{t_jZ_j - \theta X_j\right\} - (P_2 - D)\left\{t_j^2 Z_j - 2t_j\theta X_j + \theta^2 Y_j\right\} \\ &+ \left.\left. \left\{(I_j + I_{j-1})(t_j - t_{j-1})W_j\right\} \right\} \Big/ \left\{W_{j-1}\right\} \right] + \frac{hI_n^2}{2D} \end{aligned} \tag{6.19}$$

Where Y_j is given by equation (6.15)

6.4.5 Restoration Cost

Restoration cost is given by

$$\mathrm{RC} = \sum_{j=1}^{n} \left[\frac{(R_0 + R_1t_j)Z_j - R_1\theta X_j}{W_{j-1}} \right] \tag{6.20}$$

6.4.6 Lost Production Cost

$$\mathrm{LPC} = \sum_{j=1}^{n} \left[\frac{C_l(P_1 - P_2)(t_jZ_j - \theta X_j)}{W_{j-1}} \right] \tag{6.21}$$

6.4.7 Quality Cost

Quality related cost is given by

$$\mathrm{QC} = \sum_{j=1}^{n} \left[\frac{\pi \alpha P_2 \left(t_j Z_j - \theta X_j \right)}{W_{j-1}} \right] \tag{6.22}$$

6.4.8 Solution Algorithm

The following algorithm is proposed to solve the model

1. Set n=1
2. Use Golden section rule to determine h_1.
3. Calculate h_j's using equation (6.12).
4. Calculate t_j's by $t_j = t_{j-1} + h_j$.
5. Calculate X_j's, Y_j's, Z_j's and W_j's using equations (6.14) - (6.17) respectively.
6. Calculate I_j's using equation(6.13).
7. Calculate production, holding, restoration, lost production and quality costs using eqns. (6.18), (6.19), (6.20), (6.21) and (6.22) respectively.
8. Calculate cycle length by equation (6.10).
9. Calculate Expected total cost per unit time using $ETC_n = \frac{E[TC]}{E[CL]}$.
10. Set n=n+1 and repeat steps (2) through (9).

11. Check if $ETC_{n+1} < ETC_n$ then repeat (10) else $n^* = n$ and $ETC^* = ETC_n$

6.5 Numerical Results

In this section, we will present a numerical example along with sensitivity analysis to show the performance of the model and compare it with the model where we do not take restoration into account. It seems appropriate to spend some effort to find the simplified form of the model without restoration for weibull distributed time to shift for comparative purposes. The final simplified form of the model discussed in chapters 2 and 3 for weibull distributed time to shift with the inclusion of quality cost is given by

$$
\begin{aligned}
ETC(t_P) \;=\; & \left[S + (C_1P_1 - C_2P_2)\theta\, \Gamma \left\{ 1 + \frac{1}{\beta}, \left(\frac{t_P}{\theta}\right)^{\beta} \right\} + C_2P_2t_P \left(1 - e^{-(t_P/\theta)^{\beta}}\right) \right. \\
+ \; & C_l(P_1 - P_2)t_P \left(1 - e^{-(t_P/\theta)^{\beta}}\right) - C_l(P_1 - P_2)\theta\, \Gamma \left\{ 1 + \frac{1}{\beta}, \left(\frac{t_P}{\theta}\right)^{\beta} \right\} \\
+ \; & h \left\{ \frac{(P_1 - P_2)^2 - D(P_1 - P_2)}{2D} \right\} \theta^2\, \Gamma \left\{ 1 + \frac{2}{\beta}, \left(\frac{t_P}{\theta}\right)^{\beta} \right\} \\
+ \; & h \left\{ \frac{P_2(P_1 - P_2)}{D} \right\} t_P\, \theta\, \Gamma \left\{ 1 + \frac{1}{\beta}, \left(\frac{t_P}{\theta}\right)^{\beta} \right\} \\
+ \; & h \left\{ \frac{P_2(P_2 - D)}{2D} \right\} t_P^2 \left(1 - e^{-(t_P/\theta)^{\beta}}\right) + C_1P_1t_Pe^{-(t_P/\theta)^{\beta}} \\
+ \; & \left. h \left\{ \frac{P_1(P_1 - D)}{2D} \right\} t_P^2 e^{-(t_P/\theta)^{\beta}} \right] \Bigg/ \left[\frac{P_2}{D} t_P \left(1 - e^{-(t_P/\theta)^{\beta}}\right) \right. \\
+ \; & \left. \left(\frac{P_1 - P_2}{D}\right) \theta\, \Gamma \left\{ 1 + \frac{1}{\beta}, \left(\frac{t_P}{\theta}\right)^{\beta} \right\} + \frac{P_1}{D}\, t_P\, e^{-(t_P/\theta)^{\beta}} \right]
\end{aligned}
\tag{6.23}
$$

We will use “The Golden Section Rule” along with “Integer Search” to solve the problem. The data used for this numerical example is $P_1 = 270$, $P_2 = 180$, $D = 20$, $S = 370$, $h = 2$, $C_l = 50$, $a = 0.02$, $b = 1500$, $\beta = 2$, $\theta = 0.15$, $R_0 = 5$, $R_1 = 0.015$, $\pi = 100$, $\alpha = 0.05$ and $v = 10$ unless otherwise stated.

β	θ	μ	n^*	h_1	t_n	LPC	Quality Cost	Cycle Length	ETC^*	ETC_{NR}
2	0.15	0.1329	13	0.07081	0.26496	31.96	6.39	3.46625	473.92	565.21
2.5		0.1331	11	0.08509	0.22992	31.79	6.36	3.10621	474.43	544.99
3		0.1339	10	0.09148	0.20346	27.37	5.47	2.67355	486.16	530.10
2	0.1	0.0886	15	0.05804	0.23218	47.67	9.53	2.99179	518.20	653.05
	0.2	0.1772	11	0.08115	0.28110	23.64	4.73	3.70726	451.34	517.09
	0.25	0.2216	9	0.09148	0.28930	19.11	3.82	3.83225	437.77	486.91

Table 6.1: The effect of changing β and θ on the system performance

Increasing the value of β, keeping θ fixed, increases the mean but in a highly non-linear manner. As the mean increases, the time that the system spends at higher production rate also increases, decreasing the cycle length and therefore forcing the cost to become higher. On the other hand, the effect of increasing θ while keeping β fixed is linear on the mean, the operating cost per unit time will decrease because of two reasons: firstly system will spend less time in the lower production rate state which has higher unit production cost and secondly the lost production cost will decrease as a direct consequence of increased mean time to shift.

Increasing demand will try to force the cycle length to become smaller but as the cycle length decreases, the impact of fixed setup costs becomes more, so the system

P_2/D	n*	h_1	t_n	LPC	QC	Cycle Length	ETC	ETC_{NR}
5	17	0.07476	0.31718	64.67	12.93	2.296372	793.96	989.52
3	21	0.07476	0.35066	108.81	21.76	1.522736	1265.92	1625.70
2	25	0.07871	0.38121	164.39	32.88	1.103317	1851.73	2420.94

Table 6.2: Effect of changing P_2/D on the Optimal parameters

tries to obtain optimal trade-off between setup cost on one hand and the production and holding costs on the other.

P_1/P_2	n*	h_1	t_n	LPC	QC	Cycle Length	ETC	ETC_{NR}
5	17	0.0580	0.22951	47.38	1.18	2.958176	516.15	677.59
3	16	0.0580	0.23932	44.94	2.25	3.091918	505.23	649.48
1.5	13	0.0708	0.26496	31.96	6.39	3.466247	473.92	565.21

Table 6.3: Effect of changing P_1/P_2 on the Optimal parameters

Increasing P_2 decreases the unit production cost as unit production cost is convex in nature and is minimum for $P = 270$ also as the difference between P_1 and P_2 decreases the lost production cost decreases. On the other hand, increasing P_2 results in an increase in the quality related cost, but the impact of this cost is far less than that of lost production cost, as a result the Expected total cost per unit time and higher production times (to satisfy given demand).

As we increase the restoration cost parameters, the expected total cost per unit time increases and for optimal performance the production time will decreases to minimize the effect of high restoration costs.

R_0	R_1	n*	h_1	t_n	LPC	QC	cycle length	ETC
10	0.15	12	0.07081	0.25533	31.82	6.36	3.37755	470.91
50	1	13	0.07081	0.22394	31.31	6.26	2.93136	506.49
100	2.5	7	0.07081	0.20029	30.82	6.16	2.62312	538.28
200	5	6	0.06443	0.17046	25.43	5.09	2.24418	593.30
300	10	4	0.06443	0.14407	24.71	4.94	1.89802	639.31

Table 6.4: Effect of changing Restoration cost parameters on the model performance

6.6 Summary

In this chapter, we developed a mathematical model for random time to shift which included the possibility of inspection and restoration, the model involved setup, production, holding, inspection, lost production and restoration costs. In comparison, the model performed much better than the model without restoration. The reason is attributed to the fact that restoration will mainly limit the lost production cost by bringing the system back to original production rate.

Chapter 7

Stochastic time to shift with Imperfect Preventive Maintenance

7.1 Introduction

In this chapter, we will extend the model we developed in the last chapter to include Preventive maintenance and also possibility of producing bad quality components. We will consider imperfect preventive maintenance, the purpose of which will be to reduce the effective age of the system, in fact, such an activity will bring the system to some where in between “As good as new” and “As bad as old”.

7.2 Assumptions

In developing the model the following assumptions were made:

1. The process begins with a production rate P_1.
2. After time t, which is a random variable following general distribution, the process shifts to a lower production rate P_2 and also after the shift a certain fraction α of the total components produced are of sub-standard quality.
3. Demand is deterministic and constant with $P_1 \geq P_2 \geq D$.
4. Inspections are performed at time periods where integrated hazard rates are equal.
5. Cycle is repeated after every T time units.
6. The unit production cost is assumed to be convex in the production rate.
7. After a certain time h_j, where $j = 1, 2, \ldots n$ an inspection is done to determine whether the process is producing at full capacity or not, if the process is found to be operating at P_2, then an instantaneous restoration is done to bring the system back to P_1.
8. The preventive maintenance is assumed to be imperfect and it does not bring the system to as good as new state rather it decreases the age of the system by a factor b_k.

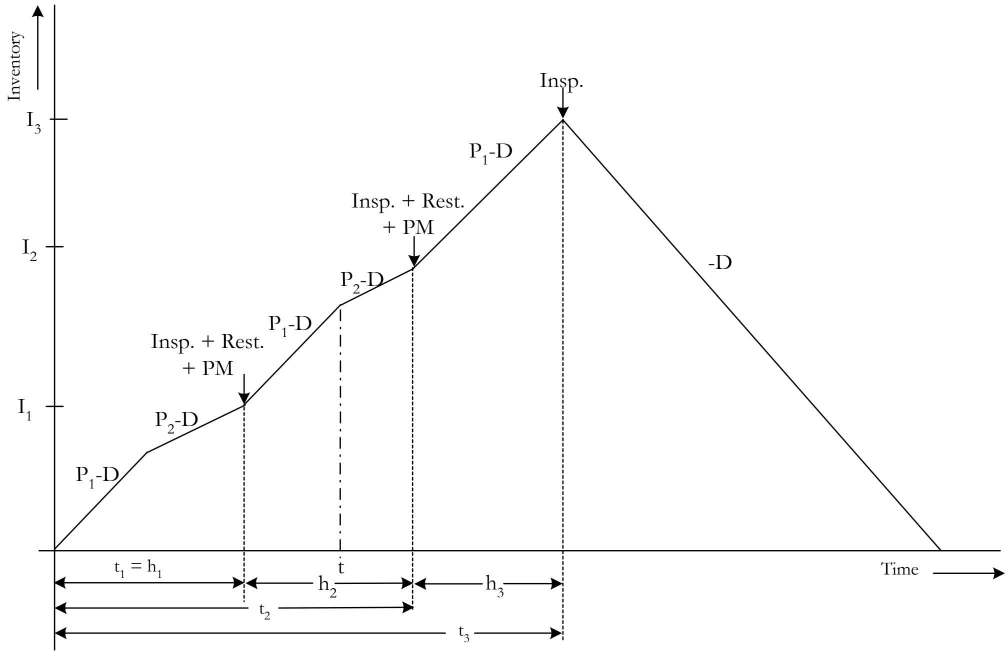

Figure 7.1: Inventory profile with varying production rate, restoration and P M for $n = 3$

7.3 Model Development

7.3.1 Setup Cost

This includes the one time cost incurred to prepare the facility for production

$$\text{Setup Cost} = S \tag{7.1}$$

7.3.2 Inspection Cost

A total of n inspections are carried out during the production stage of each cycle.

$$\text{Inspection Cost} = nv \tag{7.2}$$

7.3.3 Preventive Maintenance Cost

The PM will not be done at the end of the production cycle, so the PM cost will be

$$\text{Preventive Maintenance Cost} = (n-1)C_{pm} \tag{7.3}$$

To understand the effect of preventive maintenance on the age of the system, consider the following case. Assume y_k is the age of the system immediately before the k^{th} PM, then we assume that the PM reduces the age of the system by an amount $(1-b_k)y_k$ i.e. $w_k = b_k y_k$ (as in figure 7.2) where $0 = b_0 < b_1 < ... < b_n < 1$ denotes the effective age of the system right before (after) the k^{th} PM. So, in general,

$$y_j = w_{j-1} + h_j \tag{7.4}$$

$$w_j = b_j \times y_j \tag{7.5}$$

$$b_j = \frac{j}{j+1} \tag{7.6}$$

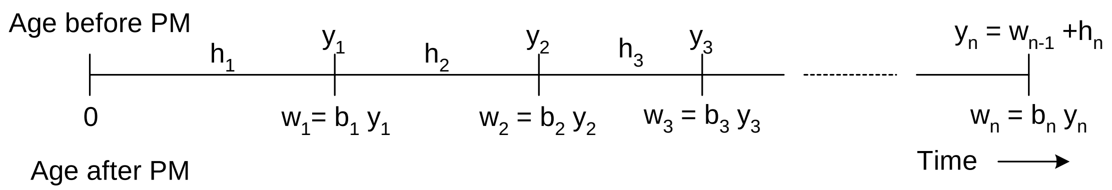

Figure 7.2: The effect of Imperfect PM on the effective age of the system

7.3.4 Production Cost

Referring to figure 7.1, and our assumption of convex unit production cost leads to the following cost equation

$$\begin{aligned}\text{Production Cost} &= \sum_{j=1}^{n}\left[\frac{\int_{w_{j-1}}^{y_j}\{C_1P_1(t-w_{j-1})+C_2P_2(y_j-t)\}\,f(t)\,dt}{\overline{F}(w_{j-1})}\right.\\ &+ \left.\frac{\int_{y_j}^{\infty}C_1P_1(y_j-w_{j-1})f(t)\,dt}{\overline{F}(w_{j-1})}\right] \qquad (7.7)\end{aligned}$$

Where $C_1 = aP_1 + \frac{b}{P_1}$ and $C_2 = aP_2 + \frac{b}{P_2}$.

7.3.5 Holding Cost

From figure 7.1 , the area under the inventory curve is given by

$$\sum_{j=1}^{n}\left\{\frac{(I_{j-1}+I)}{2}(t-w_{j-1}) \;+\; \frac{(I+I_j)}{2}(y_j-t)\right\} \;+\; \frac{I_n^2}{2D}$$

so, the holding cost will be,

$$\begin{aligned}\text{H C} &= \sum_{j=1}^{n}\left[\left(\frac{h}{2}\right)\frac{\int_{w_{j-1}}^{y_j}[(I_{j-1}+I)(t-w_{j-1}) \;+\; (I+I_j)(y_j-t)]\,f(t)\,dt}{\overline{F}(w_{j-1})}\right.\\ &+ \left.\left(\frac{h}{2}\right)\frac{\int_{y_j}^{\infty}(I_j+I_{j-1})(y_j-w_{j-1})f(t)\,dt}{\overline{F}(w_{j-1})}\right] + \frac{hI_n^2}{2D} \qquad (7.8)\end{aligned}$$

where I is the inventory level at the time to shift t. It is given by

$$\begin{aligned}I_j &= I_{j-1} + \frac{\int_{w_{j-1}}^{y_j}[(P_1-D)(t-w_{j-1}) \;+\; (P_2-D)(y_j-t)]\,f(t)\,dt}{\overline{F}(w_{j-1})}\\ &+ \frac{\int_{w_{j-1}}^{y_j}(P_1-D)(y_j-w_{j-1})f(t)\,dt}{\overline{F}(w_{j-1})} \qquad (7.9)\end{aligned}$$

The above equation (7.9) is significant as it provides a relationship between subsequent Inventory levels, also $I_0 = 0$.

7.3.6 Restoration Cost

It is the cost incurred in bringing the production rate back to the original value, i.e. P_1. Restoration cost has two components, fixed and time dependent. The time dependent part increases with the time that the system spends after the shift until being diagnosed.

$$\text{Restoration Cost} = \sum_{j=1}^{n} \left[\frac{\int_{w_{j-1}}^{y_j} \left[R_0 + R_1 \left(y_j - t\right)\right] f(t)\, dt}{\overline{F}(w_{j-1})} \right] \tag{7.10}$$

7.3.7 Lost Production Cost

The cost incurred if the producer fails to deliver the promised quantity of goods. It results from the loss of opportunity and also because of possible good will erosion.

$$\text{Lost Production Cost} = \sum_{j=1}^{n} \left[\frac{\int_{w_{j-1}}^{y_j} C_l \left(P_1 - P_2\right) \left(y_j - t\right) f(t)\, dt}{\overline{F}(w_{j-1})} \right] \tag{7.11}$$

7.3.8 Quality Cost

The cost incurred if the goods produced are of sub-standard quality.

$$\text{Quality Cost} = \sum_{j=1}^{n} \left[\frac{\int_{w_{j-1}}^{y_j} \pi \alpha \, P_2 \left(y_j - t\right) f(t)\, dt}{\overline{F}(w_{j-1})} \right] \tag{7.12}$$

The total expected cost per cycle is then

$$\begin{aligned} E[TC] &= \text{Setup cost } + \text{Inspection cost } + \text{P M cost } + \text{Production cost} \\ &+ \text{Holding cost} + \text{Restoration cost } + \text{Lost Production cost } + \text{Quality cost} \end{aligned}$$

Substituting values in above equation, we have

$$\begin{aligned} E[TC] &= S + nv + (n-1)C_{pm} \\ &+ \sum_{j=1}^{n} \left[\frac{\int_{w_{j-1}}^{y_j} \{C_1 P_1 (t - w_{j-1}) + C_2 P_2 (y_j - t)\} f(t)\, dt + \int_{y_j}^{\infty} C_1 P_1 (y_j - w_{j-1}) f(t)\, dt}{\overline{F}(w_{j-1})} \right] \\ &+ \sum_{j=1}^{n} \left[\left(\frac{h}{2}\right) \frac{\int_{w_{j-1}}^{y_j} [(I_{j-1} + I)(t - w_{j-1}) \ + \ (I + I_j)(y_j - t)] f(t)\, dt}{\overline{F}(w_{j-1})} \right. \\ &+ \left. \left(\frac{h}{2}\right) \frac{\int_{y_j}^{\infty} (I_j + I_{j-1})(y_j - w_{j-1}) f(t)\, dt}{\overline{F}(w_{j-1})} \right] + \frac{h I_n^2}{2D} \\ &+ \sum_{j=1}^{n} \left[\frac{\int_{w_{j-1}}^{y_j} [R_0 \ + R_1 (y_j - t)] f(t)\, dt}{\overline{F}(w_{j-1})} \right] + \sum_{j=1}^{n} \left[\frac{\int_{w_{j-1}}^{y_j} C_l (P_1 - P_2)(y_j - t) f(t)\, dt}{\overline{F}(w_{j-1})} \right] \\ &+ \sum_{j=1}^{n} \left[\frac{\int_{w_{j-1}}^{y_j} \pi\alpha P_2 (y_j - t) f(t)\, dt}{\overline{F}(w_{j-1})} \right] \end{aligned} \tag{7.13}$$

7.3.9 Cycle Length

Refer fig 7.1 , the cycle length is given by

$$E[CL] = t_n + \frac{I_n}{D} \tag{7.14}$$

where $t_n = t_{n-1} + h_n$. All the h_j's are related by the equal integrated hazard rate criterion [30], which states that the intervals are such that the integral of the hazard rate is always equal, i.e.,

$$\int_0^{y_1} r(t)\, dt = \int_{w_1}^{y_2} r(t)\, dt = \ldots = \int_{w_{n-1}}^{y_n} r(t)\, dt$$

7.3.10 Expected Total Cost per unit time

Using the renewal reward theorem, Expected total cost per unit time or ETC is given by

$$ETC = \frac{E[TC]}{E[CL]}$$

Where $E[TC]$ and $E[CL]$ are given by equations (7.13) and (7.14) respectively.

7.4 Weibull Distributed time to shift

In this section, we will present a numerical example using Weibull distributed time to shift, for which probability density function is given by

$$f(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} e^{-(t/\theta)^{\beta}}$$

7.4.1 Interval Estimation

Using equal integrated hazard rate, as for weibull distribution, hazard rate is given by

$$r(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}$$

Consecutively solving equation (7.3.9), we have

$$h_j = \left[w_{j-1}^{\beta} + h_1^{\beta}\right]^{1/\beta} - w_{j-1} \tag{7.15}$$

and then $t_j = t_{j-1} + h_j$, where $h_0 = t_0 = 0$.

7.4.2 Inventory Position

$$\begin{aligned}
I_j &= I_{j-1} + \frac{\int_{w_{j-1}}^{y_j} [(P_1 - D)(t - w_{j-1}) + (P_2 - D)(y_j - t)]\, f(t)\, dt}{\overline{F}(w_{j-1})} \\
&+ \frac{\int_{y_j}^{\infty} (P_1 - D)(y_j - w_{j-1}) f(t)\, dt}{\overline{F}(w_{j-1})}
\end{aligned}$$

$$\begin{aligned}
I_j &= I_{j-1} + \frac{\int_{w_{j-1}}^{y_j} [(P_1 - D)(t - w_{j-1}) + (P_2 - D)(y_j - t)]\, \frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{\beta-1} e^{-(t/\theta)^\beta}\, dt}{\overline{F}(w_{j-1})} \\
&+ \frac{\int_{y_j}^{\infty} (P_1 - D)(y_j - w_{j-1}) \frac{\beta}{\theta} \left(\frac{t}{\theta}\right)^{\beta-1} e^{-(t/\theta)^\beta}\, dt}{\overline{F}(w_{j-1})}
\end{aligned}$$

Solving above integral using the substitution $u = \left(\frac{t}{\theta}\right)^\beta$, we end up with

$$\begin{aligned}
I_j &= I_{j-1} + \Bigg[\Bigg\{ (P_1 - P_2)\theta\, X_j - [(P_1 - D)\, w_{j-1} - (P_2 - D)\, y_j]\, Z_j \\
&+ (P_1 - D)(y_j - w_{j-1})\overline{F}(y_j) \Bigg\} \Big/ \Bigg\{ \overline{F}(w_{j-1}) \Bigg\}\Bigg] \qquad (7.16)
\end{aligned}$$

Where X_j, Y_j, Z_j, $\overline{F}(w_{j-1})$ and $\overline{F}(y_j)$ are given by

$$X_j = \Gamma \left\{ 1 + \frac{1}{\beta}, \left(\frac{w_{j-1}}{\theta}\right)^\beta, \left(\frac{y_j}{\theta}\right)^\beta \right\} \qquad (7.17)$$

$$Y_j = \Gamma \left\{ 1 + \frac{2}{\beta}, \left(\frac{w_{j-1}}{\theta}\right)^\beta, \left(\frac{y_j}{\theta}\right)^\beta \right\} \qquad (7.18)$$

$$Z_j = e^{-(w_{j-1}/\theta)^\beta} - e^{-(y_j/\theta)^\beta} \qquad (7.19)$$

$$\overline{F}(w_{j-1}) = e^{-(w_{j-1}/\theta)^\beta} \qquad (7.20)$$

$$\overline{F}(y_j) = e^{-(y_j/\theta)^\beta} \qquad (7.21)$$

The incomplete gamma function $\Gamma\{a, b, c\}$ is defined as

$$\Gamma\{a, b, c\} = \int_b^c x^{a-1} e^{-x}\, dx$$

7.4.3 Production cost

Using similar procedure as described in section 7.4.2, we have

$$\text{PC} = \sum_{j=1}^{n} \frac{\left[\theta X_j(C_1P_1 - C_2P_2) - (C_1P_1w_{j-1} - C_2P_2y_j)Z_j + C_1P_1(y_j - w_{j-1})\overline{F}(y_j)\right]}{\overline{F}(w_{j-1})} \tag{7.22}$$

7.4.4 Holding Cost

$$\begin{aligned} \text{HC} &= \sum_{j=1}^{n} \left[\left(\frac{h}{2}\right) \left\{ 2I_{j-1}\left\{\theta X_j - w_{j-1}Z_j\right\} + (P_1 - D)\left\{\theta^2 Y_j - 2w_{j-1}\theta X_j + w_{j-1}^2 Z_j\right\} \right.\right. \\ &+ \ 2I_j\left\{y_jZ_j - \theta X_j\right\} - (P_2 - D)\left\{y_j^2 Z_j - 2y_j\theta X_j + \theta^2 Y_j\right\} \\ &+ \left.\left. \left\{(I_j + I_{j-1})(y_j - w_{j-1})\overline{F}(y_j)\right\} \right\} \Big/ \left\{\overline{F}(w_{j-1}\right\} \right] + \frac{hI_n^2}{2D} \end{aligned} \tag{7.23}$$

Where Y_j is given by equation (7.18)

7.4.5 Restoration Cost

Restoration cost is given by

$$\text{RC} = \sum_{j=1}^{n} \left[\frac{(R_0 + R_1y_j)\,Z_j - R_1\theta X_j}{\overline{F}(w_{j-1})} \right] \tag{7.24}$$

7.4.6 Lost Production Cost

$$\text{LPC} = \sum_{j=1}^{n} \left[\frac{C_l(P_1 - P_2)(y_jZ_j - \theta X_j)}{\overline{F}(w_{j-1})} \right] \tag{7.25}$$

7.4.7 Quality Cost

Quality related cost is given by

$$\mathrm{QC} = \sum_{j=1}^{n} \left[\frac{\pi \alpha P_2 \left(y_j Z_j - \theta X_j \right)}{\overline{F}(w_{j-1})} \right] \tag{7.26}$$

7.4.8 Solution Algorithm

The following algorithm is proposed to solve the model

1. Set n=1
2. Use Golden section rule to determine h_1.
3. Calculate h_j's using equation (7.15).
4. Calculate t_j's by $t_j = t_{j-1} + h_j$.
5. Calculate y_j, w_j and b_j by using equations (7.4) - (7.6) respectively.
6. Calculate X_j's, Y_j's, Z_j's, $\overline{F}(w_j)$ and $\overline{F}(y_j)$ using equations (7.17) - (7.21) respectively.
7. Calculate I_j's using equation(7.16).
8. Calculate production, holding, restoration, lost production and quality costs using eqns. (7.22), (7.23), (7.24), (7.25) and (7.26) respectively.
9. Calculate cycle length by eqn. (7.14).

10. Calculate Expected total cost per unit time using $ETC_n = \frac{E[TC]}{E[CL]}$.

11. Set n=n+1 and repeat steps (2) through (10).

12. Check if $ETC_{n+1} < ETC_n$ then repeat (11) else $n^* = n$ and $ETC^* = ETC_n$

7.5 Numerical Results

In this section, we will present a numerical example along with sensitivity analysis to show the performance of the model and compare it with the model where we do not perform preventive maintenance. For the sake of brevity the equations from chapter 6 are not reproduced here, the reader is referred to equations (6.9) and (6.10).

Similar to the chapter 6, "Golden Section Rule" along with "Integer Search for number of inspections" to find the solution using the algorithm presented in the preceding section. The data used for this numerical example is $P_1 = 270$, $P_2 = 180$, $D = 20$, $S = 370$, $h = 2$, $C_l = 50$, $a = 0.02$, $b = 1500$, $\beta = 2$, $\theta = 0.15$, $R_0 = 20$, $R_1 = 0.015$, $\pi = 100$, $\alpha = 0.05$, $v = 10$ and $C_{pm} = 15$ unless otherwise stated.

It is evident from table 7.1 that increasing β while keeping the θ fixed, decreases the span of the distribution, and the probability of a failure close to θ increases and so the PM model performs better, although because of increase in cycle length, some of the costs like quality cost and lost production costs have increased (the increase in latter is also indebted to the fact that unit production cost is more for P_2 and as the system spends more time at P_2, the lost production cost increases). Clearly

β	θ	μ	PM	n	h_1	t_n	y_n	PM	Rest.	LPC	QC	Cyc	ETC	Savings
2	0.15	0.1329	No	12	0.0708	0.245376			14.94075	31.68621	6.337241	3.210835	485.2552	
			Yes	7	0.0748	0.270852	0.126366	25.46	8.71	33.86	6.77	3.536765	484.97	0.2852
2.5	0.15	0.1331	No	10	0.0843	0.211663			15.19372	30.56461	6.112922	2.772706	492.9363	
			Yes	6	0.0811	0.242833	0.112221	23.49513	7.280245	26.97202	5.394404	3.192151	477.0277	15.9086
3	0.15	0.1339	No	8	0.0941	0.188117			14.16074	28.59653	5.719306	2.468974	496.6157	
			Yes	6	0.0851	0.24041	0.10795	23.6047	6.302143	21.46567	4.293134	3.177334	469.7127	26.903

Table 7.1: The effect of changing β on the system performance

β	θ	μ	PM	n	h_1	t_n	y_n	PM	Rest.	LPC	QC	Cyc	ETC	Savings
2	0.1	0.0886	No	14	0.056452	0.211226			28.00021	44.91778	8.983557	2.728965	540.0344	
			Yes	8	0.058044	0.228073	0.103629	35.65976	15.54243	45.67575	9.135151	2.944495	534.0995	5.9349
2	0.2	0.1772	No	10	0.083553	0.264219			9.201628	24.72815	4.94563	3.480874	458.3318	1.694
			Yes	6	0.085093	0.280419	0.135289	20.29994	5.378134	24.64766	4.929531	3.694592	460.0258	
2	0.25	0.2216	No	9	0.093399	0.280197			6.32157	19.75775	3.95155	3.709364	442.5277	2.3487
			Yes	5	0.097868	0.287887	0.145162	15.75655	3.731359	20.62444	4.124887	3.807941	444.8764	

Table 7.2: The effect of changing θ on the system performance

restoration cost is much less in the case when PM is done, it is because of the age reduction by the PM.

If we increase θ keeping β fixed, as shown in table 7.2, then the probability of having a shift inside the production time decreases and in such a case, the additional amount spent in PM activity is not warranted as we are unlikely to have a shift, for this reason the model without PM performs better.

The reason for lower n in PM case is two folds: firstly because of higher costs associated with each inspection (both inspection and PM are done) and secondly because of age reduction through PM activity is associated with the number of inspections in the model with PM, so it always results in lower n^*. Increasing demand (table 7.3) forces the cycle length to shorten and hence the effect of restoration cost is enhanced resulting in better performance of the PM model. On the other hand, reverse effect is seen upon increasing P_2, cycle length increases and so the effect of high costs due to cost production and quality forces the cost of the PM model to become higher than the model without PM.

If we increase the restoration cost parameters, given that the PM decreases the effective age of the system and also the probability of shift, the restoration cost will always be lower for the PM case. So for higher restoration costs, the PM model out performs the Restoration only model.

As pointed out earlier in the discussion of table 7.1, with the PM the cycle length is increased because of age reduction. As a consequence, the system spends more

P2/D	PM	n	h_1	t_n	y_n	PM	Rest.	LPC	QC	Cyc	ETC	Savings
5	No	15	0.073797	0.285814			31.13111	62.49034	12.49807	2.071683	818.3055	
	Yes	9	0.078706	0.331989	0.147634	50.02898	18.06068	68.52087	13.70417	2.39861	813.6374	4.6681
3	No	18	0.075963	0.322286			58.23368	111.1279	22.22557	1.398481	1311.208	
	Yes	11	0.078706	0.374107	0.160949	92.5396	32.6648	115.7739	23.15478	1.620928	1297.043	14.165
2	No	20	0.076675	0.342902			92.80064	170.5542	34.11084	0.991142	1923.704	
	Yes	11	0.081146	0.385709	0.165941	134.9328	50.21234	184.0319	36.80637	1.111664	1896.487	27.217

Table 7.3: The effect of changing P_2/D on the system performance

P1/P2	PM	n	h_1	t_n	y_n	PM	Rest.	LPC	QC	Cyc	ETC	Savings
5	No	16	0.054994	0.219975			14.21241	48.67055	1.216764	2.831837	526.9893	1.0675
	Yes	9	0.058043	0.244832	0.108876	38.20693	7.969793	52.35712	1.308928	3.140791	528.0568	
3	No	15	0.058368	0.226057			14.43591	45.13814	2.256907	2.919973	516.3213	0.6731
	Yes	8	0.061989	0.243576	0.110673	33.49182	8.012495	48.86199	2.443099	3.135094	516.9944	
1.5	No	12	0.070834	0.245376			14.94075	31.68621	6.337241	3.210835	485.2552	
	Yes	7	0.074759	0.270852	0.126366	25.44698	8.706676	33.85491	6.770981	3.536765	484.9717	0.2835

Table 7.4: The effect of changing P_1/P_2 on the system performance

R_0	R_1	PM	n	h_1	t_n	y_n	PM	Rest.	LPC	QC	Cyc	ETC	Savings
10	0.15	No	13	0.07095	0.255814			7.788314	31.9375	6.3875	3.346602	477.7179	2.9015
		Yes	7	0.074759	0.270852	0.126366	25.44698	4.35441	33.85491	6.770981	3.536765	480.6194	
50	1	No	10	0.070206	0.222012			33.83671	30.79141	6.158284	2.907636	506.4802	
		Yes	6	0.074759	0.246366	0.11886	23.30363	20.51001	33.41792	6.683584	3.218382	497.4119	9.0683
100	2.5	No	8	0.069191	0.195702			59.76268	29.47712	5.895423	2.566327	538.1812	
		Yes	6	0.070814	0.233363	0.112587	24.52353	39.21157	30.12064	6.024127	3.058287	517.2953	20.8859
200	5	No	6	0.067108	0.16438			100.7848	27.16518	5.433037	2.160443	592.5306	
		Yes	5	0.068375	0.201131	0.101417	22.70922	71.04151	27.69239	5.538477	2.642099	554.9828	37.5478
300	10	No	4	0.070646	0.141292			128.8315	28.86712	5.773425	1.853928	639.1394	
		Yes	6	0.061989	0.204283	0.098557	27.82897	104.91	23.29663	4.659327	2.695033	590.1025	49.0369

Table 7.5: The effect of changing restoration cost parameters on the system performance

α	PM	n	h_1	t_n	y_n	PM	Rest.	LPC	QC	Cyc	ETC	Savings
0	No	11	0.075084	0.249026			15.01508	35.23672	0	3.247416	478.5462	
	Yes	6	0.078708	0.259377	0.125137	22.2084	8.552223	36.86311	0	3.3771	477.8951	0.6511
0.05	No	12	0.070834	0.245376			14.94075	31.68621	6.337241	3.210835	485.2552	
	Yes	7	0.074759	0.270852	0.126366	25.44698	8.706676	33.85491	6.770981	3.536765	484.9717	0.2835
0.1	No	12	0.068615	0.237689			14.54239	29.80525	11.9221	3.11593	491.3889	
	Yes	7	0.070814	0.256558	0.119697	26.77799	8.321941	30.51414	12.20566	3.360969	491.1537	0.2352
0.15	No	13	0.065328	0.235543			14.51181	27.24154	16.34493	3.095498	497.0171	0.0294
	Yes	7	0.068375	0.247722	0.115575	27.67958	8.078362	28.52499	17.11499	3.251494	497.0465	
0.2	No	14	0.062754	0.234802			14.54192	25.30933	20.24747	3.091583	502.3156	0.4359
	Yes	7	0.068375	0.247722	0.115575	27.67958	8.078362	28.52499	22.81999	3.251494	502.7515	
0.25	No	14	0.061246	0.229161			14.23376	24.14359	24.14359	3.020747	507.2555	0.6691
	Yes	7	0.064428	0.233423	0.108904	29.28687	7.675153	25.43197	25.43197	3.07305	507.9246	
0.3	No	14	0.059959	0.224346			13.9684	23.16792	27.80151	2.960088	511.9762	0.5421
	Yes	8	0.061989	0.243576	0.110673	32.69309	7.82141	23.84835	28.61803	3.211688	512.5183	

Table 7.6: The effect of changing α on the system performance

time after the shift until being restored, because of this reason, the quality costs are higher in the PM model. So, in accordance with our discussion, increasing the percentage of bad items forces the gap between the quality costs of the two models to increase and thereby resulting in better performance of the restoration only model.

7.6 Conclusions

In this chapter, we extended the mathematical model developed in the last chapter to include the possibility of performing Preventive maintenance, and to make it more realistic, we consider age reduction by the PM (successive PM's reduce the age by a reducing factor). Later, we presented a numerical example to illustrate the model's performance against the one presented in the last chapter. The results show clearly that for higher values of β, demand, restoration costs the model performs better than the one with restoration only.

Chapter 8

Random production rate after the shift

8.1 Introduction

In this chapter, we will relax the assumption that the production rate after the shift is deterministic and known apriori, instead we will assume it to be a random variable following general distribution. This approach will bring the model closer to the real case, e.g. in the slippage case, the belt creep sets in slowly and the production rate could be any value less than the original production rate. Later, in the chapter, we will analyze the model for the presence of local minima and present numerical examples for certain well known distribution to gauge the performance of the model.

8.2 Assumptions

In developing the model the following assumptions were made:

1. The process begins with a production rate P_1.

2. After time t, which is random and assumed to be exponentially distributed $0 \leq t < \infty$, the process shifts to a lower production rate P_2.

3. The production rate after the shift is assumed to follow a general distribution between P_1 and P_2 with p.d.f. $g(P)$.

4. Demand is deterministic and constant with $P_1 \geq P_2 \geq D$.

5. Cycle is repeated after every T time units.

6. Unit production cost is assumed convex in production rate.

8.3 Development of the Model

In this section we will develop the model for the case where the production rate after the shift is uniformly distributed between two levels P_1 and P_2 rather than being constant. The model will be developed as:

$$\begin{aligned} E[TC] &= \int_0^{t_P} \int_{P_2}^{P_1} TC_1\, g(P) f(t)\, dP\, dt \\ &+ \int_{t_P}^{\infty} \int_{P_2}^{P_1} TC_2\, g(P) f(t)\, dP\, dt \end{aligned}$$

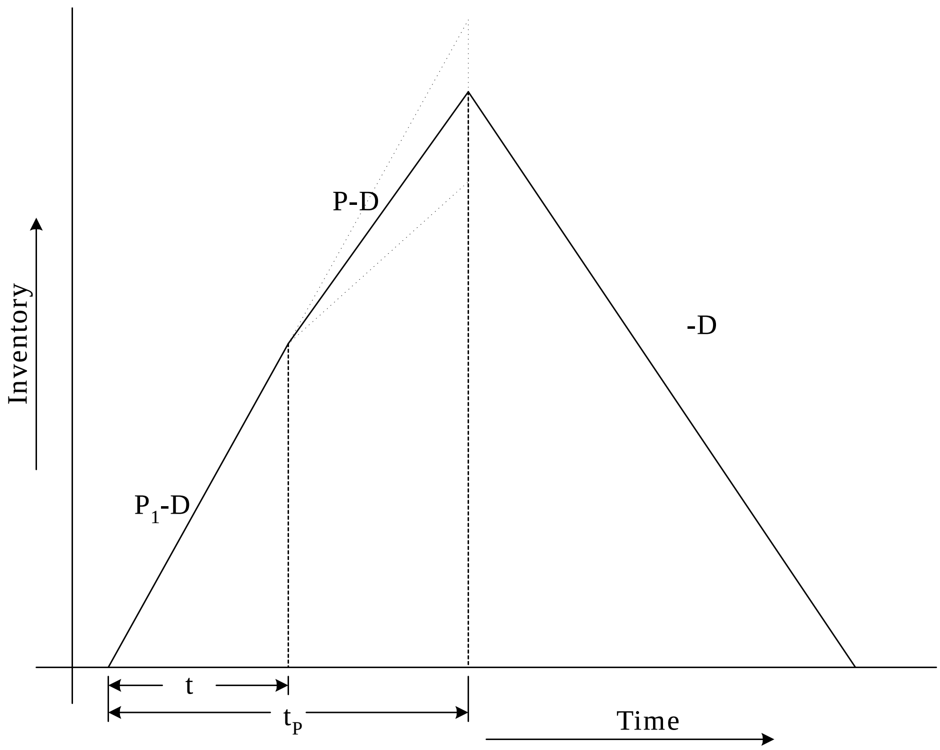

Figure 8.1: Inventory profile with random production rate after the shift

8.3.1 Uniform Distribution

If production rate after the shift (P) follows uniform distribution between P_1 and P_2, i.e. $g(P) = \frac{1}{P_1 - P_2}$ and time to shift follows exponential distribution, i.e. $f(t) = \lambda e^{-\lambda t}$, then we have

$$
\begin{aligned}
E[TC] &= \int_0^{t_P} \int_{P_2}^{P_1} TC_1 \frac{1}{P_1 - P_2} dP \ \lambda e^{-\lambda t} dt \\
&+ \int_{t_P}^{\infty} \int_{P_2}^{P_1} TC_2 \frac{1}{P_1 - P_2} dP \ \lambda e^{-\lambda t} dt
\end{aligned}
$$

Where TC_1 and TC_2 are the same as equations (4.5) and (4.10), which we reproduce here for convenience.

$$\begin{aligned}TC_1 &= S + \left(aP_1 + \frac{b}{P_1}\right) P_1 t + \left(aP + \frac{b}{P}\right) P(t_P - t) \\ &+ h\Bigg[t^2 \left\{\frac{(P_1 - P)^2 - D(P_1 - P)}{2D}\right\} + t_P t \left\{\frac{P(P_1 - P)}{D}\right\} \\ &+ t_P^2 \left\{\frac{P(P - D)}{2D}\right\}\Bigg] + C_l(P_1 - P)(t_P - t)\end{aligned}$$

and

$$TC_2 = S + t_P \left(aP_1^2 + b\right) + h \left\{\frac{P_1(P_1 - D)t_P^2}{2D}\right\}$$

Similarly

$$\begin{aligned}E[CL] &= \int_0^{t_P} \int_{P_2}^{P_1} CL_1 \frac{1}{P_1 - P_2} dP \; \lambda e^{-\lambda t} dt \\ &+ \int_{t_P}^{\infty} \int_{P_2}^{P_1} CL_2 \frac{1}{P_1 - P_2} dP \; \lambda e^{-\lambda t} dt\end{aligned}$$

Where $CL_1 = \frac{Pt_P + (P_1 - P)t}{D}$ and $CL_2 = \frac{P_1}{D} t_P$ are the same as equations (4.6) and (4.11). Then Expected Total Cost per unit time will be given by

$$ETC(t_P) = \frac{E[TC]}{E[CL]}$$

Upon substitution, we get

$$\begin{aligned} ETC(t_P) &= \Bigg[S + CP_1 \left(\frac{1}{\lambda} - \frac{e^{-\lambda t_P}}{\lambda} \right) + \left\{ \frac{a\left(P_1^2 \, + \, P_1\, P_2 \, + \, P_2^2\right) + 3b}{3} \right\} \left\{ t_P - \frac{1}{\lambda} + \frac{e^{-\lambda t_P}}{\lambda} \right\} \\ &+ \; h \left\{ \frac{(P_1 - P_2)\left\{2(P_1 - P_2) - 3D\right\}}{12D} \right\} \left\{ \frac{2}{\lambda^2}\left(1 - e^{-\lambda t_P}\right) - \frac{2}{\lambda} t_P e^{-\lambda t_P} - t_P^2 e^{-\lambda t_P} \right\} \\ &+ \; \left\{ \frac{h t_P (P_1 - P_2)(P_1 + 2P_2)(1 - e^{-\lambda t_P} - \lambda t_P e^{-\lambda t_P}}{6\lambda D} \right\} + \frac{h P_1 (P_1 - D) t_P^2}{2D} \, e^{-\lambda t_P} \\ &+ \; h t_P^2 (1 - e^{-\lambda t_P}) \left\{ \frac{2(P_1^2 + P_1 P_2 + P_2^2) - 3D(P_1 + P_2)}{12D} \right\} \\ &+ \; C_l (P_1 - P_2) \frac{\lambda t_P - 1 + e^{-\lambda t_P}}{2\lambda} \Bigg] / \\ & \left[\frac{P_1}{\lambda D}(1 - e^{-\lambda t_P}) + \left(\frac{P_1 + P_2}{2D} \right) \left\{ t_P - \frac{1}{\lambda}(1 - e^{-\lambda t_P}) \right\} \right] \end{aligned} \tag{8.1}$$

8.3.2 Model analysis

Let

$$\psi(t_P) = (CL)\ (TC') - (TC)\ (CL') \tag{8.2}$$

In the analysis of the model we will use the definitions we stated in equation (8.2)

Using simple substitution, it is trivial to show that

$$\lim_{t_P \to 0} \psi(t_P) = -\frac{SP_1}{D}$$

For $\lim\limits_{t_P \to \infty} \psi(t_P)$, the terms that have Exponential part will grow to zero as $t \to \infty$

So, we are left with only those terms that does not involve exponential part. i.e.

$$\begin{aligned}
\lim_{t_P \to \infty} \psi(t_P) &= -\frac{S\,P_1}{2\,D} + \frac{b\,P_1}{D\,\lambda} + \frac{h\,{P_1}^2}{4\,D\,\lambda^2} - \frac{C_1\,{P_1}^2}{2\,D\,\lambda} + \frac{C_l\,{P_1}^2}{2\,D\,\lambda} - \frac{h\,{P_1}^3}{12\,D^2\,\lambda^2} + \frac{a\,{P_1}^3}{3\,D\,\lambda} \\
&- \frac{S\,P_2}{2\,D} - \frac{C_1\,P_1\,P_2}{2\,D\,\lambda} - \frac{C_l\,P_1\,P_2}{2\,D\,\lambda} + \frac{h\,{P_1}^2\,P_2}{6\,D^2\,\lambda^2} + \frac{a\,{P_1}^2\,P_2}{3\,D\,\lambda} - \frac{h\,{P_2}^2}{4\,D\,\lambda^2} \\
&- \frac{h\,P_1\,{P_2}^2}{12\,D^2\,\lambda^2} + \frac{a\,P_1\,{P_2}^2}{3\,D\,\lambda} + t_P^2\Big\{- \frac{h\,P_1^2}{8\,D} + \frac{h\,{P_1}^3}{12\,D^2} - \frac{h\,P_1\,P_2}{4\,D} + \frac{h\,{P_1}^2\,P_2}{6\,D^2} \\
&- \frac{h\,{P_2}^2}{8\,D} + \frac{h\,P_1\,{P_2}^2}{6\,D^2} + \frac{h\,{P_2}^3}{12\,D^2}\Big\} + t_P\left\{-\frac{h\,{P_1}^2}{4\,D\,\lambda} + \frac{h\,{P_1}^3}{6\,D^2\,\lambda} + \frac{h\,{P_2}^2}{4\,D\,\lambda} - \frac{h\,{P_2}^3}{6\,D^2\,\lambda}\right\}
\end{aligned}$$

In order to show that $\lim_{t_P \to \infty} \psi(t_P)$ is positive, we just need to show that the co-efficient of t_P^2 is positive. i.e.,

$$\begin{aligned}
&= \left\{- \frac{h\,P_1^2}{8\,D} + \frac{h\,{P_1}^3}{12\,D^2} - \frac{h\,P_1\,P_2}{4\,D} + \frac{h\,{P_1}^2\,P_2}{6\,D^2} - \frac{h\,{P_2}^2}{8\,D} + \frac{h\,P_1\,{P_2}^2}{6\,D^2} + \frac{h\,{P_2}^3}{12\,D^2}\right\} > 0 \\
&= \left\{\frac{-\left[h\,(P_1+P_2)\,\left(3\,D\,P_1 - 2\,{P_1}^2 + 3\,D\,P_2 - 2\,P_1\,P_2 - 2\,{P_2}^2\right)\right]}{24\,D^2}\right\} > 0 \\
&= \left\{\frac{\left[h\,(P_1+P_2)\,\left(2(\,{P_1}^2 +\,P_1\,P_2 +\,{P_2}^2) - 3\,D\,(P_1+P_2)\right)\right]}{24\,D^2}\right\} > 0
\end{aligned}$$

For the above equation to be positive, $2(\,{P_1}^2 +\,P_1\,P_2 +\,{P_2}^2) - 3\,D\,(P_1+P_2)$ must be positive.

$$\begin{aligned}
&= 2(\,{P_1}^2 +\,P_1\,P_2 +\,{P_2}^2) - 3\,D\,(P_1+P_2) > 0 \\
&= (P_1+P_2)^2 +\,(P_1^2+P_2^2) - 3\,D\,(P_1+P_2) > 0
\end{aligned}$$

$$\begin{aligned}
&= (P_1+P_2)^2 +\,(P_1^2+P_2^2) > 3\,D\,(P_1+P_2) \\
&= (P_1+P_2)(P_1+P_2) +\,(P_1^2+P_2^2) > 2\,D\,(P_1+P_2) + D\,(P_1+P_2)
\end{aligned}$$

Since $P_1 > P_2 > D$, so we have $(P_1 + P_2)(P_1 + P_2) > 2D(P_1 + P_2)$, and also $P_1^2 + P_2^2 > DP_1 + DP_2$, now combining both of these parts, we have

$$(P_1 + P_2)^2 + (P_1^2 + P_2^2) > 3\,D\,(P_1 + P_2)$$

$$\lim_{t_P \to \infty} \psi(t_P) \to \infty$$

So, the function has at least one local minimum. To further emphasize our point, we would just show empirically that the function behaves in a convex manner by plotting Expected cost per unit time and time of production.

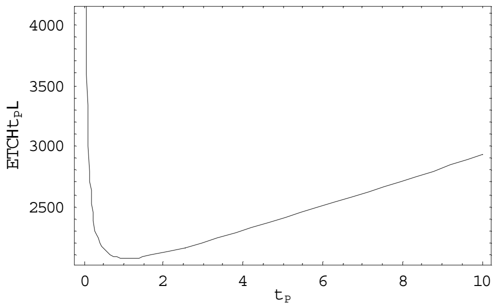

Figure 8.2: Plot of Expected Total Cost per unit time for uniformly distributed production rate

Numerical Example

A numerical example along with sensitivity analysis will be presented to gauge the performance of the model. We will use "The Golden Section Rule" to solve the problem. The data used for this numerical example is $P_1 = 270, P_2 = 180, D = 20, S = 370, h = 2, a = 0.02, b = 1500, C_l = 20$ and $\lambda = 10$ unless otherwise stated.

λ	t_P	Setup	Production	Holding	Lost Production	Cycle Length	ETC^*
0.1	0.3268985	84.07	219.18	81.53	1.08	4.401237	385.86
1	0.3190672	88.01	219.78	78.02	9.82	4.204145	395.63
5	0.3063977	97.38	221.51	70.62	35.44	3.799727	424.95
10	0.3190672	97.23	222.69	70.59	52.79	3.805248	443.30
20	0.3522425	90.79	223.64	75.44	66.76	4.07513	456.63

Table 8.1: Effect of changing λ on the Optimal parameters

As we increase λ, the mean time to shift decreases and less time is available for P_1, but as the per unit production cost is lower for P_1 than P_2, so logically, the time of production should decrease to decrease the overall cost, but as λ is increased further, the portion of time during which P_1 is available becomes even smaller, in order to meet the demand the production time t_P starts to increase. All the time increase of λ will prompt an increase in Expected total cost per unit time because of higher unit production cost of P_2.

P_1/P_2	t_P	Setup	Production	Holding	Lost Production	Cycle Length	ETC^*
5	0.2273644	159.07	238.51	45.63	127.83	2.326066	571.04
4	0.2400348	148.86	236.45	48.30	121.47	2.485634	555.08
3	0.2653788	131.83	233.24	53.77	110.57	2.806737	529.41
2	0.299001	110.66	227.15	62.72	82.22	3.343465	482.75
1.5	0.3190672	97.23	222.69	70.59	52.79	3.805248	443.30

Table 8.2: Effect of changing P_1/P_2 on the Optimal parameters

Increasing P_2 decreases the unit production cost as unit production cost is convex in nature resulting in decr5ease in production cost, also increasing P_2 forces the lost production cost to decrease as less quantity is resulted in lost production. These factors combined causes the expected cost per unit time to decrease.

P_2/D	t_P	Setup	Production	Holding	Lost Production	Cycle Length	ETC^*
5	0.4469258	126.85	401.68	90.32	107.40	2.916854	726.25
3	0.6128218	155.93	670.52	107.82	194.59	2.372918	1128.86
2.5	0.6996855	164.52	805.07	114.21	240.02	2.24896	1323.82
2	8.35E-01	173.02	1006.98	120.65	309.51	2.138538	1610.16
1.5	1.100854	176.06	1343.72	124.81	428.61	2.1016	2073.19

Table 8.3: Effect of changing P_2/D on the Optimal parameters

Increasing demand forces the cycle length to decrease because of quicker consumption, resulting in higher setup cost. As a consequence, the system attempts to increase the cycle length to nullify the effect of setup costs, which results in higher holding, production costs.

8.3.3 Triangular Distribution

If P_2 follows triangular distribution with parameters D, and P_1, with P_2 the most likely value or the mode of the distribution. Then the probability density function is given by

$$g(P) = \begin{cases} \frac{2(P-D)}{(P_1-D)(P_2-D)} & \text{if } D \leq P \leq P_2 \\ \frac{2(P_1-P)}{(P_1-D)(P_1-P_2)} & \text{if } P_2 < P \leq P_1 \\ 0 & \text{otherwise} \end{cases}$$

The time to shift follows exponential distribution, i.e. $f(t) = \lambda e^{-\lambda t}$, then we have

$$\begin{aligned} E[TC] &= \int_0^{t_P} \int_D^{P_2} TC_1 \frac{2(P-D)}{(P_1-D)(P_2-D)} dP \; \lambda e^{-\lambda t} dt + \int_0^{t_P} \int_{P_2}^{P_1} TC_1 \frac{2(P_1-P)}{(P_1-D)(P_1-P_2)} dP \; \lambda e^{-\lambda t} dt \\ &+ \int_{t_P}^{\infty} \int_D^{P_2} TC_2 \frac{2(P-D)}{(P_1-D)(P_2-D)} dP \; \lambda e^{-\lambda t} dt + \int_{t_P}^{\infty} \int_{P_2}^{P_1} TC_2 \frac{2(P_1-P)}{(P_1-D)(P_1-P_2)} dP \; \lambda e^{-\lambda t} dt \end{aligned}$$

Where TC_1 and TC_2 are the same as equations (4.5) and (4.10). Similarly,

$$\begin{aligned} E[CL] &= \int_0^{t_P}\int_D^{P_2} CL_1 \frac{2(P-D)}{(P_1-D)(P_2-D)} dP\ \lambda e^{-\lambda t} dt + \int_0^{t_P}\int_{P_2}^{P_1} CL_1 \frac{2(P_1-P)}{(P_1-D)(P_1-P_2)} dP\ \lambda e^{-\lambda t} dt \\ &+ \int_{t_P}^{\infty}\int_D^{P_2} CL_2 \frac{2(P-D)}{(P_1-D)(P_2-D)} dP\ \lambda e^{-\lambda t} dt + \int_{t_P}^{\infty}\int_{P_2}^{P_1} CL_2 \frac{2(P_1-P)}{(P_1-D)(P_1-P_2)} dP\ \lambda e^{-\lambda t} dt \end{aligned}$$

Where CL_1 and CL_2 are the same as equations (4.6) and (4.11). Then Expected Total Cost per unit time will be given by

$$ETC(t_P) = \frac{E[TC]}{E[CL]}$$

Which upon simplification, yields

$$\begin{aligned} ETC(t_P) &= \left[S + C_1 P_1 \left\{ \frac{1}{\lambda} - \frac{e^{-\lambda t_P}}{\lambda} \right\} + \left\{ \frac{\left\{ e^{-\lambda t_P} - 1 + \lambda t_P \right\} C_l \left(2P_1 - P_2 - D\right)}{3\lambda} \right\} \right. \\ &+ \left\{ \frac{h\left\{2(1-e^{-\lambda t_P}) - \lambda t_P e^{-\lambda t_P}(2+\lambda t_P)\right\}\left[3(P_1^2 + D^2) - 3P_2(P_1 - D) + P_2^2 - 7P_1 D\right]}{12D\lambda^2} \right\} \\ &+ \left\{ \frac{\left\{1 - e^{-\lambda t_P} - \lambda t_P e^{-\lambda t_P}\right\} h t_P \left[(P_1^2 - D^2) + (P_1 - P_2)(P_2 + D)\right]}{6D\lambda} \right\} \\ &+ \left\{ \frac{\left(1 - e^{-\lambda t_P}\right) h t_P^2 \left[(P_1^2 - D^2) + (P_1 + P_2)(P_2 - D)\right]}{12D} \right\} + h \left\{ \frac{P_1(P_1 - D)}{2D} t_P^2 \right\} e^{-\lambda t_P} \\ &+ \left. \left\{ \frac{\left\{e^{-\lambda t_P} - 1 + \lambda t_P\right\}\left[6b + aD^2 + a\left\{P_1^2 + P_1(P_1 + P_2)(P_1 + D)\right\}\right]}{6\lambda} \right\} \right] \Bigg/ \\ & \left[\frac{\left(-1 + \lambda t_p + e^{-\lambda t_P}\right)(P_2 + D) + P_1\left\{2 + \lambda t_P - 2e^{-\lambda t_P}\right\}}{3D\lambda} \right] \end{aligned} \tag{8.3}$$

8.3.4 Model analysis

In the analysis of the model we will use the definitions we stated in equation (8.2)

Using simple substitution, it is trivial to show that

$$\lim_{t_P \to 0} \psi(t_P) = -\frac{SP_1}{D}$$

For $\lim_{t_P \to \infty} \psi(t_P)$, the terms that have Exponential part will grow to zero as $t \to \infty$

So, we are left with only those terms that does not involve exponential part. i.e.

$$\begin{aligned}
\lim_{t_P \to \infty} \psi(t_P) &= -\frac{S}{3} - \frac{D\,h}{9\,\lambda^2} - \frac{S\,P_1}{3\,D} + \frac{h\,P_1}{18\,\lambda^2} + \frac{b\,P_1}{D\,\lambda} + \frac{a\,D\,P_1}{6\,\lambda} - \frac{C_1\,P_1}{3\,\lambda} - \frac{C_l\,P_1}{3\,\lambda} \\
&+ \frac{5\,h\,{P_1}^2}{18\,D\,\lambda^2} + \frac{a\,{P_1}^2}{6\,\lambda} - \frac{C_1\,{P_1}^2}{3\,D\,\lambda} + \frac{2\,C_l\,{P_1}^2}{3\,D\,\lambda} - \frac{h\,{P_1}^3}{18\,D^2\,\lambda^2} + \frac{a\,{P_1}^3}{6\,D\,\lambda} - \frac{S\,P_2}{3\,D} \\
&- \frac{2\,h\,P_2}{9\,\lambda^2} + \frac{h\,P_1\,P_2}{6\,D\,\lambda^2} + \frac{a\,P_1\,P_2}{6\,\lambda} - \frac{C_1\,P_1\,P_2}{3\,D\,\lambda} - \frac{C_l\,P_1\,P_2}{3\,D\,\lambda} + \frac{h\,{P_1}^2\,P_2}{18\,D^2\,\lambda^2} \\
&+ \frac{a\,{P_1}^2\,P_2}{6\,D\,\lambda} - \frac{h\,{P_2}^2}{9\,D\,\lambda^2} - \frac{h\,P_1\,{P_2}^2}{18\,D^2\,\lambda^2} + \frac{a\,P_1\,{P_2}^2}{6\,D\,\lambda} + t_P^2 \Bigg\{ -\frac{D\,h}{36} - \frac{hP_1}{18} \\
&+ \frac{h\,{P_1}^3}{36\,D^2} - \frac{h\,P_2}{18} - \frac{h\,P_1\,P_2}{36\,D} + \frac{h\,{P_1}^2\,P_2}{18\,D^2} + \frac{h\,P_1\,{P_2}^2}{18\,D^2} + \frac{h\,{P_2}^3}{36\,D^2} \Bigg\} \\
&+ t_P \Bigg\{ \frac{D\,h}{18\,\lambda} - \frac{h\,P_1}{18\,\lambda} - \frac{h\,{P_1}^2}{6\,D\,\lambda} + \frac{h\,{P_1}^3}{9\,D^2\,\lambda} + \frac{h\,P_2}{9\,\lambda} - \frac{h\,P_1\,P_2}{9\,D\,\lambda} + \frac{h\,{P_1}^2\,P_2}{18\,D^2\,\lambda} \\
&+ \frac{h\,P_1\,{P_2}^2}{18\,D^2\,\lambda} - \frac{h\,{P_2}^3}{18\,D^2\,\lambda} \Bigg\}
\end{aligned}$$

In order to show that $\lim_{t_P \to \infty} \psi(t_P)$ is positive, we just need to show that the co-efficient of t_P^2 is positive. i.e.,

$$\begin{aligned}
&= -\frac{D\,h}{36} - \frac{hP_1}{18} + \frac{h\,{P_1}^3}{36\,D^2} - \frac{h\,P_2}{18} - \frac{h\,P_1\,P_2}{36\,D} + \frac{h\,{P_1}^2\,P_2}{18\,D^2} + \frac{h\,P_1\,{P_2}^2}{18\,D^2} + \frac{h\,{P_2}^3}{36\,D^2} \\
&= \frac{h\,(D + P_1 + P_2)\,\left({P_1}^2 + P_1\,P_2 + {P_2}^2 - D^2 - D\,P_1 - D\,P_2\right)}{36\,D^2} > 0
\end{aligned}$$

For the above equation to be positive, ${P_1}^2 + P_1\,P_2 + {P_2}^2 - D^2 - D\,P_1 - D\,P_2$ must be positive.

$$\begin{aligned}
&= {P_1}^2 + P_1\,P_2 + {P_2}^2 - D^2 - D\,P_1 - D\,P_2 > 0 \\
&= (P_1^2 - D^2) + P_2(P_1 + P_2) - D\,(P_1 + P_2) > 0 \\
&= (P_1^2 - D^2) + (P_2 - D)(P_1 + P_2) > 0
\end{aligned}$$

Since $P_1 > P_2 > D$, it is trivial to show that the above equation is positive.

$$\lim_{t_P \to \infty} \psi(t_P) \to \infty$$

So, the function has at least one local minimum. To further emphasize our point, we would just show empirically that the function behaves in a convex manner by plotting Expected cost per unit time and time of production.

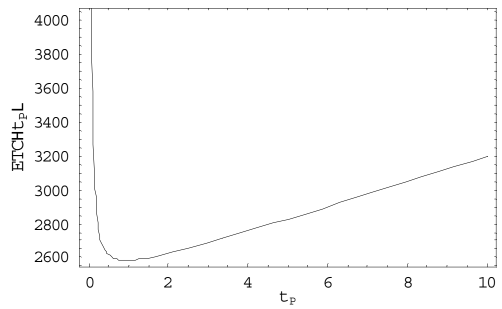

Figure 8.3: Plot of Expected Total Cost per unit time for triangular distribution

Numerical Example

We will study the model's performance specifically taking a numerical example in this section, the values that we will use are $P_1 = 270, P_2 = 180, D = 20, S = 370, h = 2, a = 0.02, b = 1500, C_l = 20,\ \lambda = 10$ and$\mu = 0.00445$ unless otherwise explicitly stated.

We expected to see an increase in the expected total cost per unit time because of convex unit production cost which means that for any production rate other than P_1 the production cost will be higher. For very high value of λ the production

λ	t_P^*	Setup	Production	Holding	Lost Production	Cycle Length	ETC^*
0.1	0.3268985	84.41	219.51	81.34	2.73	4.383179	387.99
1	0.2937257	98.85	222.55	70.46	23.74	3.743126	415.60
5	0.2322035	142.45	231.09	50.12	82.76	2.597338	506.42
10	0.2195356	166.42	238.39	43.18	133.22	2.223281	581.20
20	0.3190672	132.99	250.84	53.87	219.28	2.782213	656.98

Table 8.4: Effect of changing λ on system performance

time starts to increase because of the demand constraint which forces the system to satisfy demand at all times.

P_1/P_2	t_P^*	Setup	Production	Holding	Lost Production	Cycle Length	ETC^*
5	0.1580235	242.98	248.44	30.98	160.39	1.522732	682.79
4	0.1658523	231.43	247.64	32.32	160.19	1.598761	671.58
3	0.1736826	218.87	245.50	33.82	154.81	1.690467	653.00
2	0.1990317	188.41	242.15	38.61	147.29	1.963789	616.47
1.5	0.2195356	166.42	238.39	43.18	133.22	2.223281	581.20

Table 8.5: Effect of P_1/P_2 on optimal system performance

When we increase P_2 keeping P_1 fixed, we are actually moving on the downhill side of the convex unit production cost curve, so although we increase P_2 but we get a discounted unit production cost, this along with the fact that increasing P_2 results in lesser lost production forces the cost to go down, also the time required decreases because of relatively higher yield. The results shown in table 8.5 are in accordance

P_2/D	t_P^*	Setup	Production	Holding	Lost Production	Cycle Length	ETC^*
5	0.2732101	245.05	429.69	48.29	257.09	1.509921	980.13
3	0.3600738	312.94	713.47	53.96	444.55	1.182325	1524.93
2.5	0.4264235	318.42	854.26	58.15	541.69	1.161982	1772.52
2	0.5748099	296.17	1063.20	66.94	684.57	1.249301	2110.88
1.5	0.9554588	234.26	1401.02	82.76	866.58	1.579472	2584.63

Table 8.6: Effect of changing P_2/D on system performance

with the ones previously obtained for similar situations for other models, increasing demand forces the cycle length to decrease and so the system is forced to increase the production time to achieve optimal trade-off between different costs.

8.3.5 Exponential Distribution

If P_2 follows exponential distribution with parameters μ, but it can take any value between P_1, with P_2 only, so we have a case of truncated exponential distribution. In this case, the probability density function is given by

$$g(P) = \mu e^{-\mu P}$$

The time to shift follows exponential distribution, i.e. $f(t) = \lambda e^{-\lambda t}$, then we have

$$E[TC] = \frac{\int_0^{t_P} \int_{P_2}^{P_1} TC_1 \mu e^{-\mu P} dP \ \lambda e^{-\lambda t} dt}{\int_{P_2}^{P_1} \mu e^{-\mu P} dP} + \frac{\int_{t_P}^{\infty} \int_{P_2}^{P_1} TC_2 \mu e^{-\mu P} dP \ \lambda e^{-\lambda t} dt}{\int_{P_2}^{P_1} \mu e^{-\mu P} dP}$$

Where TC_1 and TC_2 are the same as equations (4.5) and (4.10). Similarly,

$$E[CL] = \frac{\int_0^{t_P} \int_{P_2}^{P_1} CL_1 \mu e^{-\mu P} dP \ \lambda e^{-\lambda t} dt}{\int_{P_2}^{P_1} \mu e^{-\mu P} dP} + \frac{\int_{t_P}^{\infty} \int_{P_2}^{P_1} CL_2 \mu e^{-\mu P} dP \ \lambda e^{-\lambda t} dt}{\int_{P_2}^{P_1} \mu e^{-\mu P} dP}$$

Where CL_1 and CL_2 are the same as equations (4.6) and (4.11). Then Expected Total Cost per unit time will be given by

$$ETC(t_P) = \frac{E[TC]}{E[CL]}$$

Which upon simplification, yields

$$\begin{aligned}
ETC \quad = \quad & \Bigg[S + C_1 P_1 \left\{ \frac{1}{\lambda} - \frac{e^{-\lambda t_P}}{\lambda} \right\} + h \frac{P_1(P_1 - D)}{2D} t_P^2 e^{-\lambda t_P} + \Bigg\{ \left(\frac{-1 + e^{-\lambda t_P} + \lambda t_P}{(e^{\mu\, P_1} - e^{\mu\, P_2})\, \lambda\, \mu^2} \right) \times \\
& \left[e^{\mu\, P_1}\ \{2\,a + b\,\mu^2 + a\,\mu\, P_2\ (2 + \mu\, P_2)\} - e^{\mu\, P_2}\ \{2\,a + b\,\mu^2 + a\,\mu\, P_1\ (2 + \mu\, P_1)\} \right] \Bigg\} \\
+ \quad & \Bigg\{ \left(\frac{h\,\{2(1 - e^{-\lambda t_P}) - e^{-\lambda t_P} \lambda t_P (2 + \lambda t_P)\}}{2D\lambda^2\mu^2(e^{\mu\, P_1} - e^{\mu\, P_2})} \right) \times \Big[- e^{\mu P_2}\,(2 + D\mu) + e^{\mu\, P_1}\,\{2 + D\mu \\
+ \quad & \mu(P_1 - P_2)\,(-2 - D\mu + \mu(P_1 - P_2))\,\} \Big] \Bigg\} + \Bigg\{ \left(\frac{h t_P\,\{-1 + e^{-\lambda t_P} + \lambda t_P e^{-\lambda t_P}\}}{D\lambda\mu^2\,(e^{\mu\, P_1} - e^{\mu\, P_2})} \right) \times \\
& \left[e^{\mu\, P_1}\ \{2 - \mu P_1(1 + \mu P_2) + \mu P_2(2 + \mu P_2)\} - e^{\mu P_2}(2 + \mu P_1) \right] \Bigg\}
\end{aligned}$$

$$\begin{aligned}
+ \quad & \Bigg\{ \left(\frac{h t_P^2\,\{1 - e^{-\lambda t_P}\}}{2D\mu^2\,(e^{\mu\, P_1} - e^{\mu\, P_2})} \right) \times \left[e^{\mu\, P_1}\ \{2 - D\mu + \mu P_2(2 - D\mu + \mu P_2)\} + e^{\mu P_2}\,\{-2 + D\mu + \mu P_1(-2 + D\mu - \mu P_1)\} \right] \Bigg\} \\
+ \quad & \left\{ \frac{C_l\,\{-1 + \lambda t_P + e^{-\lambda t_P}\}\,\left[e^{\mu P_2} + e^{\mu P_1}\,\{-1 + \mu(P_1 - P_2)\}\right]}{\lambda\mu\,(e^{\mu\, P_1} - e^{\mu\, P_2})} \right\} \Bigg] \Bigg/ \Bigg[\frac{P_1}{D} t_P e^{-\lambda t_P} \\
+ \quad & \frac{\mu P_1\,\left[e^{\mu P_1}(1 - \lambda t_p e^{-\lambda t_P} - e^{-\lambda t_P}) - e^{\mu P_2} \lambda t_P (1 - e^{-\lambda t_P})\right] + \left\{-1 + \lambda t_P + e^{-\lambda t_P}\right\}\left\{e^{\mu P_1}(1 + \mu P_2) - e^{\mu P_2}\right\}}{D\mu\lambda\,(e^{\mu\, P_1} - e^{\mu\, P_2})} \Bigg]
\end{aligned} \tag{8.4}$$

8.3.6 Model analysis

In the analysis of the model we will use the definitions we stated in equation (8.2) Using simple substitution, it is trivial to show that

$$\lim_{t_P \to 0} \psi(t_P) = -\frac{SP_1}{D}$$

For $\lim_{t_P \to \infty} \psi(t_P)$, the terms that have Exponential part will grow to zero as $t \to \infty$ So, we are left with only those terms that does not involve exponential part and they can be represented as $a + bt_P + ct_P^2$, so as $t_P \to \infty$, only c retains our interest because even if the other co-efficients are negative then still because of higher rate of increase of t_P, the expression will not be effected by their values. After some algebraic manipulation, c can be written as

$$\begin{aligned} c &= h\frac{\left[\left(e^{\mu P_1} - e^{\mu P_2}\right) + \mu P_2 e^{\mu P_1}\left(1 - \frac{P_1}{P_2} e^{\mu (P_2 - P_1)}\right)\right]}{2D^2\mu^3\left(e^{\mu P_1} - e^{\mu P_2}\right)} \times \\ & \left[\frac{2\left(e^{\mu P_1} - e^{\mu P_2}\right) + \mu(2P_2 - D)e^{\mu P_1}\left(1 - \frac{(2P_1 - D)}{(2P_2 - D)} e^{\mu (P_2 - P_1)}\right)}{2D^2\mu^3\left(e^{\mu P_1} - e^{\mu P_2}\right)}\right. \\ &+ \left.\frac{\mu^2 P_2(P_2 - D)e^{\mu P_1}\left(1 - \frac{P_1(P_1 - D)}{P_2(P_2 - D)} e^{\mu (P_2 - P_1)}\right)}{2D^2\mu^3\left(e^{\mu P_1} - e^{\mu P_2}\right)}\right] \end{aligned}$$

The above expression will be positive if μ becomes large but if μ is smaller then that is the only region where, in general $xe^{1-x} > 1$; where $x > 1$. So, we find the value of c as $\mu \to 0$.

$$\lim_{\mu \to 0} c = \frac{h\,(P_1 + P_2)\,\left(2\,{P_1}^2 + (P_1 + P_2)\,(2\,P_2 - 3\,D)\right)}{24\,D^2}$$

Which has been already proved positive in section 8.3.2. Hence, we can say that

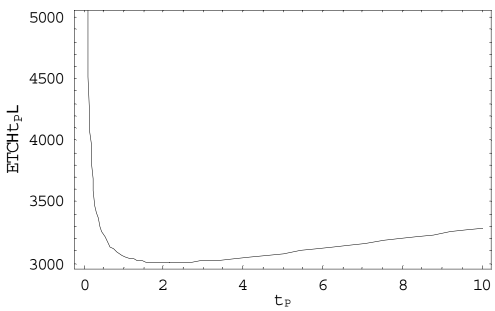

Figure 8.4: Plot of Expected Total Cost per unit time for the exponentially distributed production rate

$\lim\limits_{t_P \to \infty} \psi(t_P) \to \infty$, or that there exists at least one local minima. To further emphasize our point, we would just show empirically that the function behaves in a convex manner by plotting Expected cost per unit time and time of production.

Numerical Example

In this section, we will present a numerical example to illustrate the performance of the model under various parameter combinations. The values that we will use are $P_1 = 270, P_2 = 180, D = 20, S = 370, h = 2, a = 0.02, b = 1500, C_l = 20,\ \lambda = 10$ and $\mu = 0.00445$ unless otherwise explicitly stated.

λ	t_P	Setup	Production	Holding	Lost Production	Cycle Length	ETC^*
0.1	0.2937257	93.55	219.18	82.61	1.04	3.955045	396.38
1	0.1990332	140.02	219.58	89.06	6.74	2.642419	455.40
5	0.1375212	209.61	220.55	111.65	20.71	1.76516	562.53
10	0.1248507	237.64	221.41	134.61	33.01	1.556991	626.67
20	0.1248507	247.33	222.64	169.29	50.67	1.49598	689.93

Table 8.7: Effect of changing failure rate on the Optimal parameters

This result is in-line with our expectation, we expected to see an increase in ETC and decrease in production time. The reason is that the cycle length will decrease to hamper the increase of the lost production cost and also higher production costs, the result is an overall increase in ETC.

μ	t_P	Setup	Production	Holding	Lost Production	Cycle Length	ETC^*
0.00375	0.1200116	246.50	221.31	149.68	31.74	1.501037	649.23
0.004	0.1200116	246.57	221.33	140.74	31.86	1.500617	640.50
0.0045	0.1248507	237.65	221.42	133.19	33.04	1.556901	625.29
0.005	0.132682	224.51	221.54	131.32	34.75	1.648017	612.12
0.0055	0.1375212	217.18	221.63	126.09	35.90	1.703627	600.81

Table 8.8: Effect of changing μ on the Optimal parameters

While selecting μ, care must be taken that the value of μ is such that the mean of the distribution must lie between the extremes P_1 and P_2, for this to be true, μ can only be between $1/P_1$ and $1/P_2$. The effect of this change is the same as that of changing P_2 in the earlier cases. Increasing the mean production rate after the shift causes the holding cost to decrease which in turn forces the ETC downwards.

P_2/D	t_P	Setup	Production	Holding	Lost Production	Cycle Length	ETC^*
5	0.1658523	327.23	399.43	197.18	72.07	1.13071	995.91
3	0.206862	442.96	666.91	259.30	137.33	0.8352891	1506.50
2.5	0.2195356	502.61	800.68	277.31	170.39	0.7361545	1750.99
2	0.2527027	550.26	1002.00	326.37	229.40	0.6724132	2108.02
1.5	0.2858969	652.99	1337.30	370.73	324.64	0.5666229	2685.66

Table 8.9: Effect of changing P_2/D on the Optimal parameters

If we increase the demand while keeping the other parameters fixed, longer production time is required to get the optimal trade-off, naturally forcing the holding, production, lost production costs to go up.

8.4 Conclusions

In this chapter, we extended the mathematical model developed in chapter 4 by relaxing the assumption of the fixed production rate after the shift. Besides developing the frame work for such models, we also looked at properties of these models and took three special cases for illustrative purposes, uniform, triangular and exponential distributions. We also presented numerical examples showing the effect of changing different parameters on the optimal performance of these models.

Chapter 9

Conclusions

9.1 Summary

The main problem that this thesis discussed was Economic Production Quantity models for deteriorating processes, where due to speed losses the production rate may shift from a higher to a certain lower value at any random point in time.

With the advent of new techniques like total productive maintenance (TPM) which considers speed losses as one of the contributors to the overall equipment effectiveness, the importance of variable production rate is more than ever, there was a genuine need to develop some basic framework which could be utilized to develop mathematical models to deal with these situations.

The classical economic production lot size model assumes a constant production rate that is predetermined, inflexible and perfect quality. Recent models have relaxed the inflexible production rate assumption. Production rates in many cases such as orders filled by machine can be changed. Moreover, unit production cost and process quality depend on the production rate. Most of the models attempts to make a trade-off between the productivity losses from making too small batches and the opportunity cost of tying up capital in inventory as large batches.

9.2 Major Contributions

The major contributions of this thesis could be summarized as follows:

- We developed a mathematical model for deterministic time to shift, we also determined some formulas for determining the production time for such a case. This model helped us in developing the basic framework for the latter part of the thesis.

- In the next phase, we relaxed the assumption of deterministic time to shift and developed a more general model which accounts for random time to shift. We developed models for two specific types of production costs, namely convex production cost and two stage production cost. We showed that the model will have at least one local minima and also presented numerical examples to illustrate the model's behaviour.

- In phase three of the thesis, we developed a mathematical model for random time to shift which included the possibility of inspection and restoration, the decision variables were n and t_P. In comparison, the model performed much better than the model without restoration. The reason is attributed to the fact that restoration will mainly limit the lost production cost by bringing the system back to original production rate.

- In phase four, we added preventive maintenance to our mathematical model to go along with inspection and restoration. Also, we assumed for more realistic approach, that the preventive maintenance is imperfect. The results show clearly that for higher values of shape factor (β), demand, restoration costs the model performs better than the one with restoration only.

- In the final phase, we revisited the model developed in phase two and relaxed the assumption of the fixed production rate after the shift, considered three specific cases namely uniform, triangular and exponential distributions for the production rate after the shift.

9.3 Possible directions for future research

We have attempted to address the major and important areas of the variable production rate problem, but clearly a lot of work needs to be done to fully understand the different aspects of the problem. Some avenues for future research could be:

- Through out the thesis, we have considered the demand to be deterministic and known, the possibility of incorporating stochastic demand needs further investigation, although such a problem may result in extremely complex model.
- We have limited our study to only two levels of production, some generalized model could be developed for n different production levels with varying probability.
- Several minor setups to go along with one major setup with or with out considerable time duration could be considered. Also, we have considered instantaneous restorations, this assumption could be relaxed.
- Sampling plans (with or without errors) could be integrated as means of detecting the process capability.
- Restoration model could be considered with random production rate after the shift.
- Preventive maintenance model could be considered with random production rate after the shift.
- Residual life approach could be used with both the restoration and PM models to recalculate the inspection intervals after reaching the time of inspection.

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